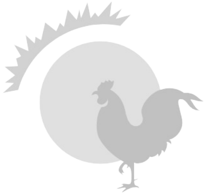
University of Applied Sciences Northwestern Switzerland
School of Business

Vague Knowledge: Fuzzy Logic



MSc Business Information Systems

Acknowledgement

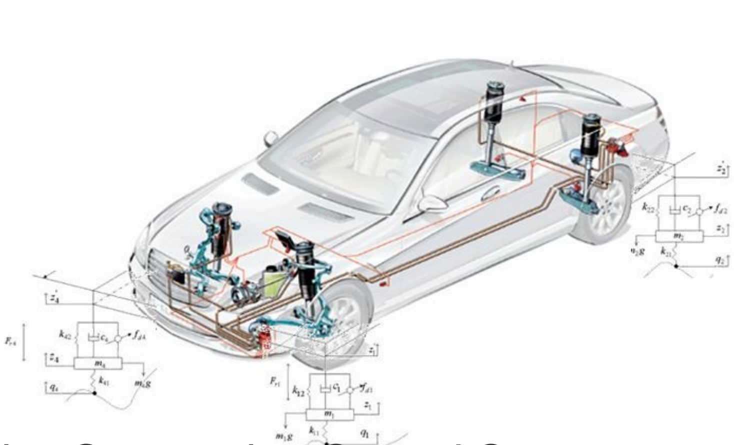
- Slides are based on slides from Prof. Dr. Knut Hinkelmann

FUZZY SETS

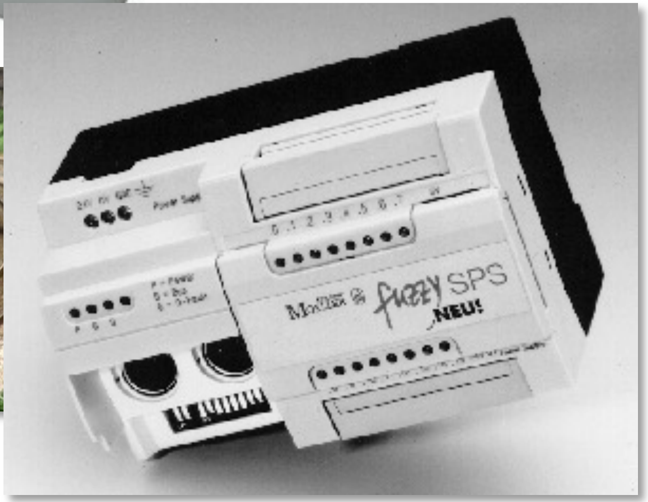
Applications of Fuzzy Logic

- Fuzzy Systems became well-known as control systems (Washing machine, …)
- Other application areas:
    - Diagnosis
    - Language understanding

Washing Machine



Active Suspension Control System



Inventor of Fuzzy Logic

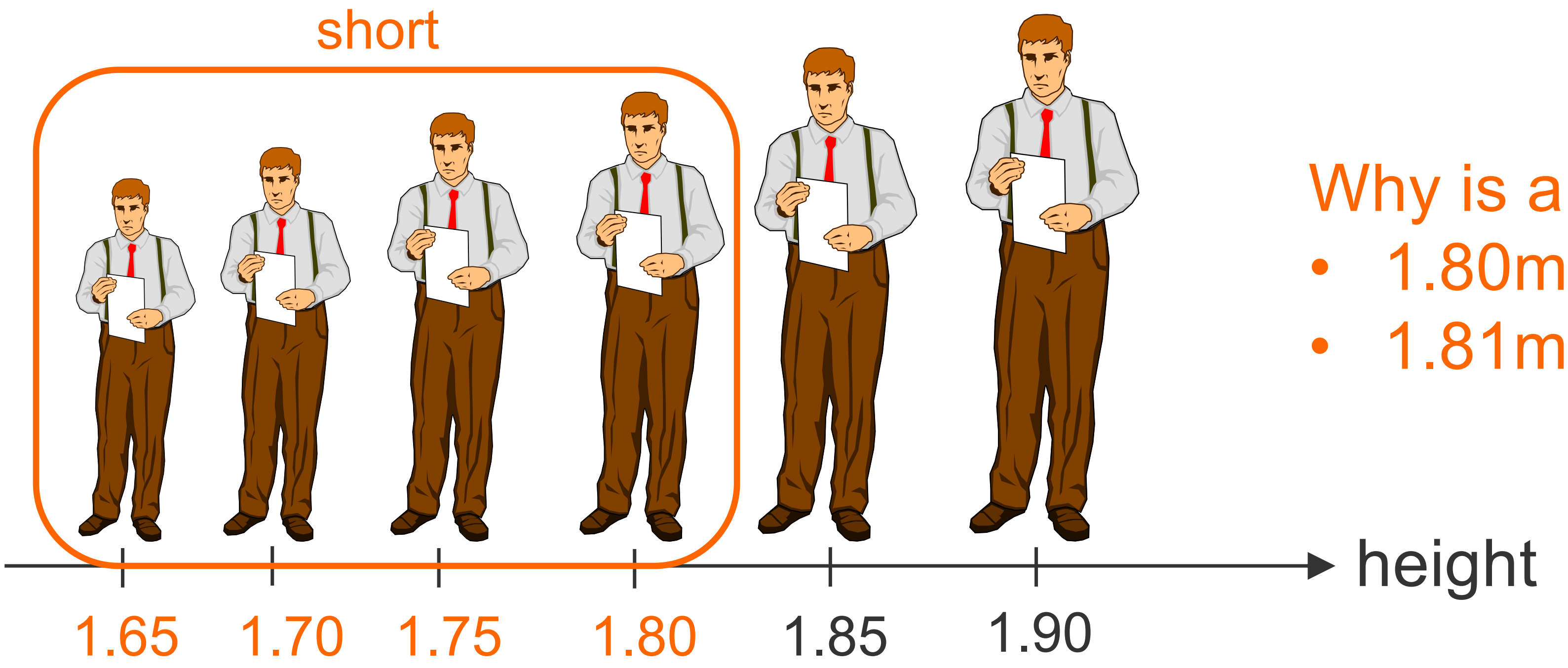
Lotfi Zadeh 2010

Lotfi Zadeh 1945

Who is short and who is tall? And who is medium?

Classical vs. Fuzzy Sets

- When is a man short?
- Classical Set Theory: Either short or not short.
  E.g.: set of short men S= {m|height(m) ≤ 1.80}

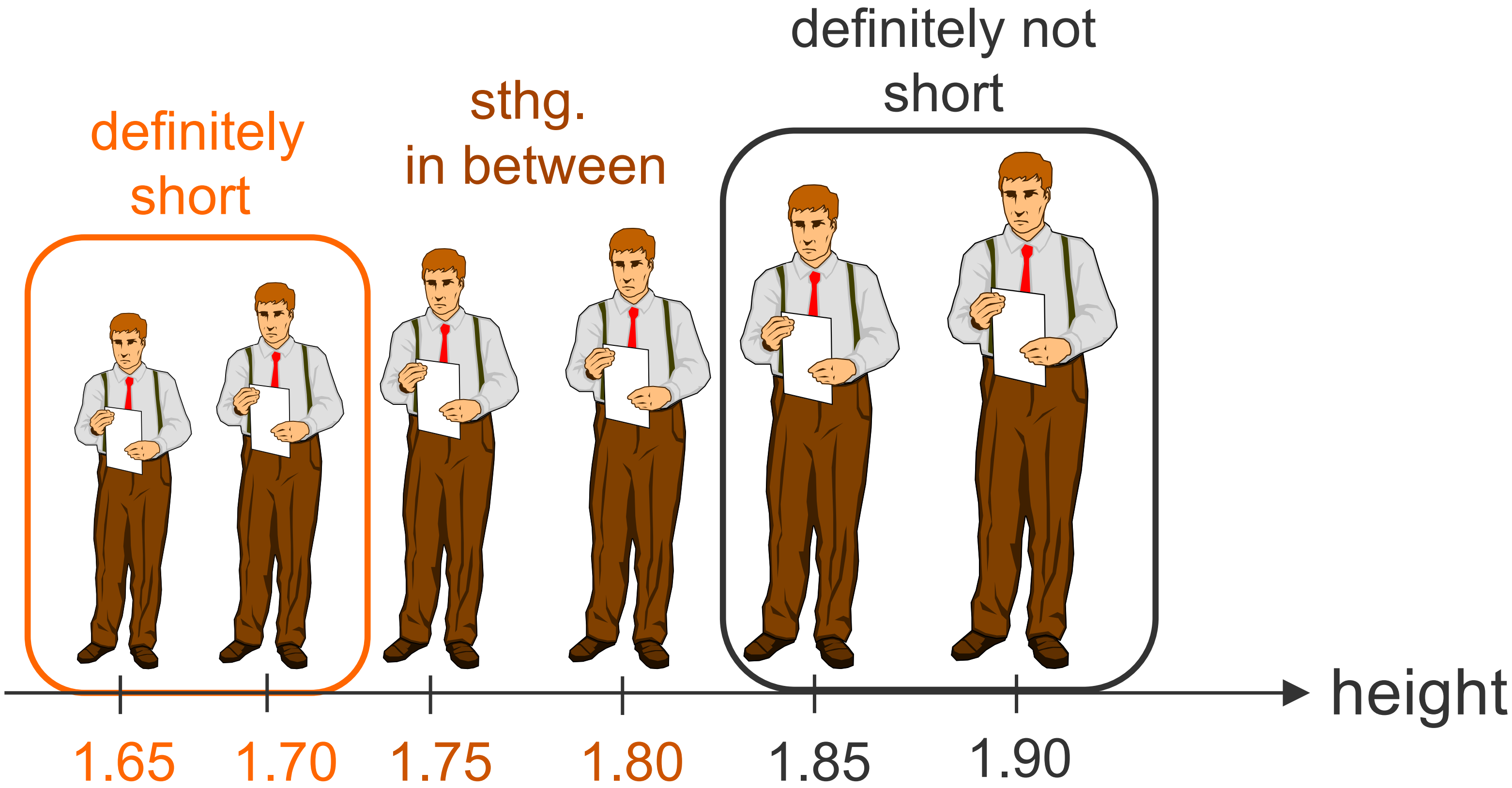


Why is a man of

- 1.80m short
- 1.81m not short?

Classical vs. Fuzzy Sets

- Fuzzy sets have unsharp boundaries:

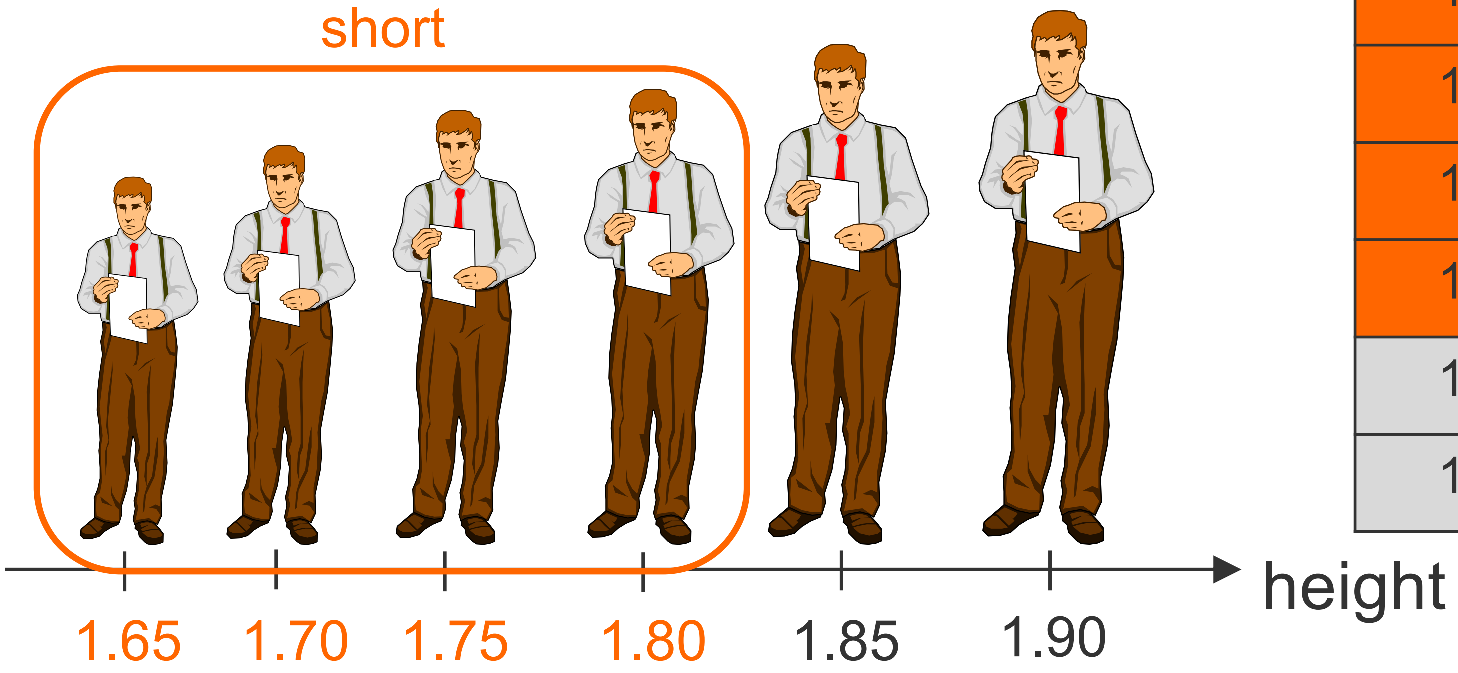


Classical vs. Fuzzy Sets

- A classical set can be seen as a special case of a fuzzy set, where the fuzziness of the set boudary is infinitely small.
- Classical sets are also called crisp sets.

Representation by membership function

Classical sets, e.g.: set of short men $S= \{m|height(m) \leq 1.80\}$

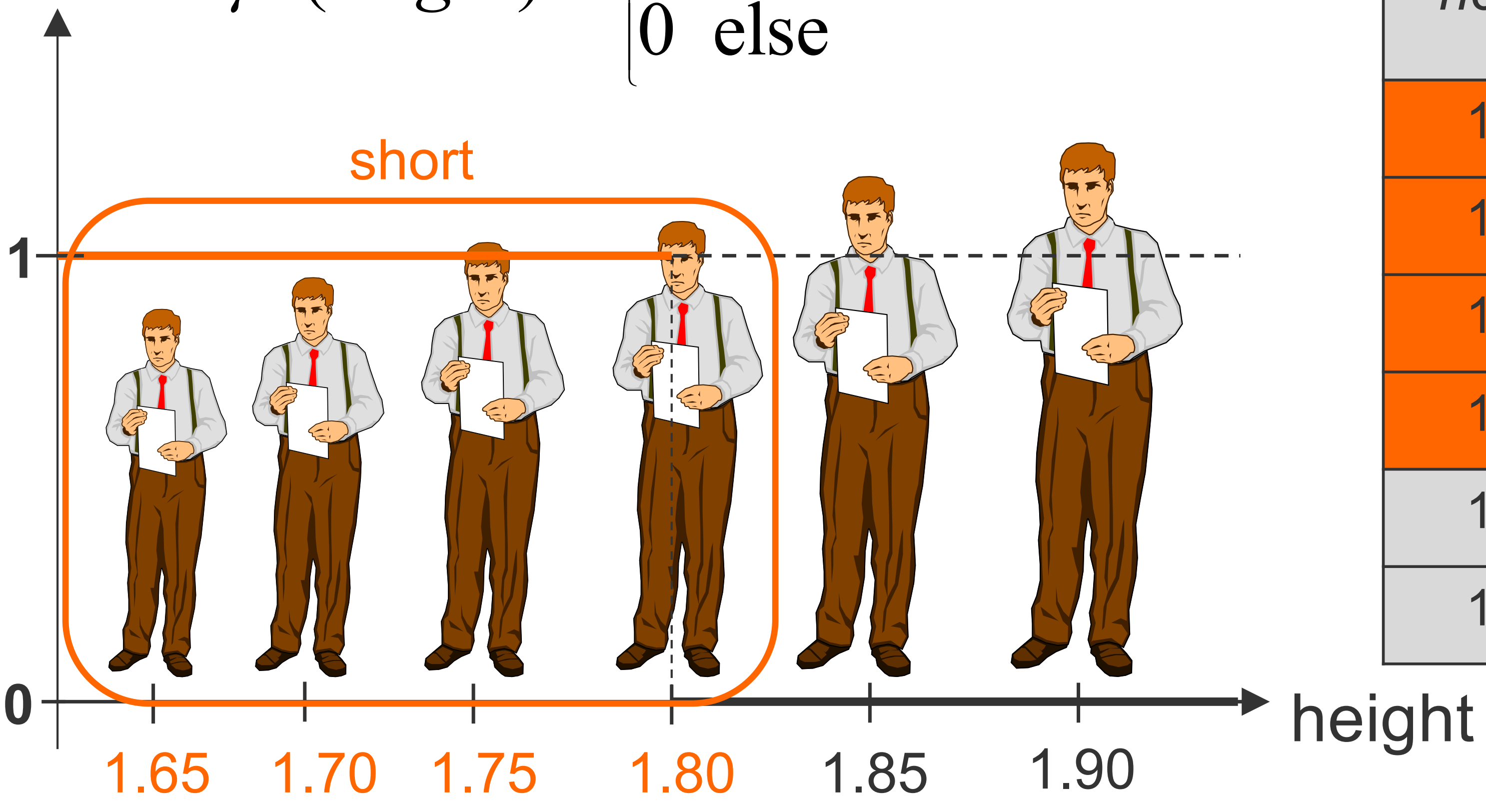


height	*short?*
1.65	1
1.70	1
1.75	1
1.80	1
1.85	0
1.90	0

Representation by membership function

Classical sets, e.g.: set of short men S= {m|height(m) ≤ 1.80}

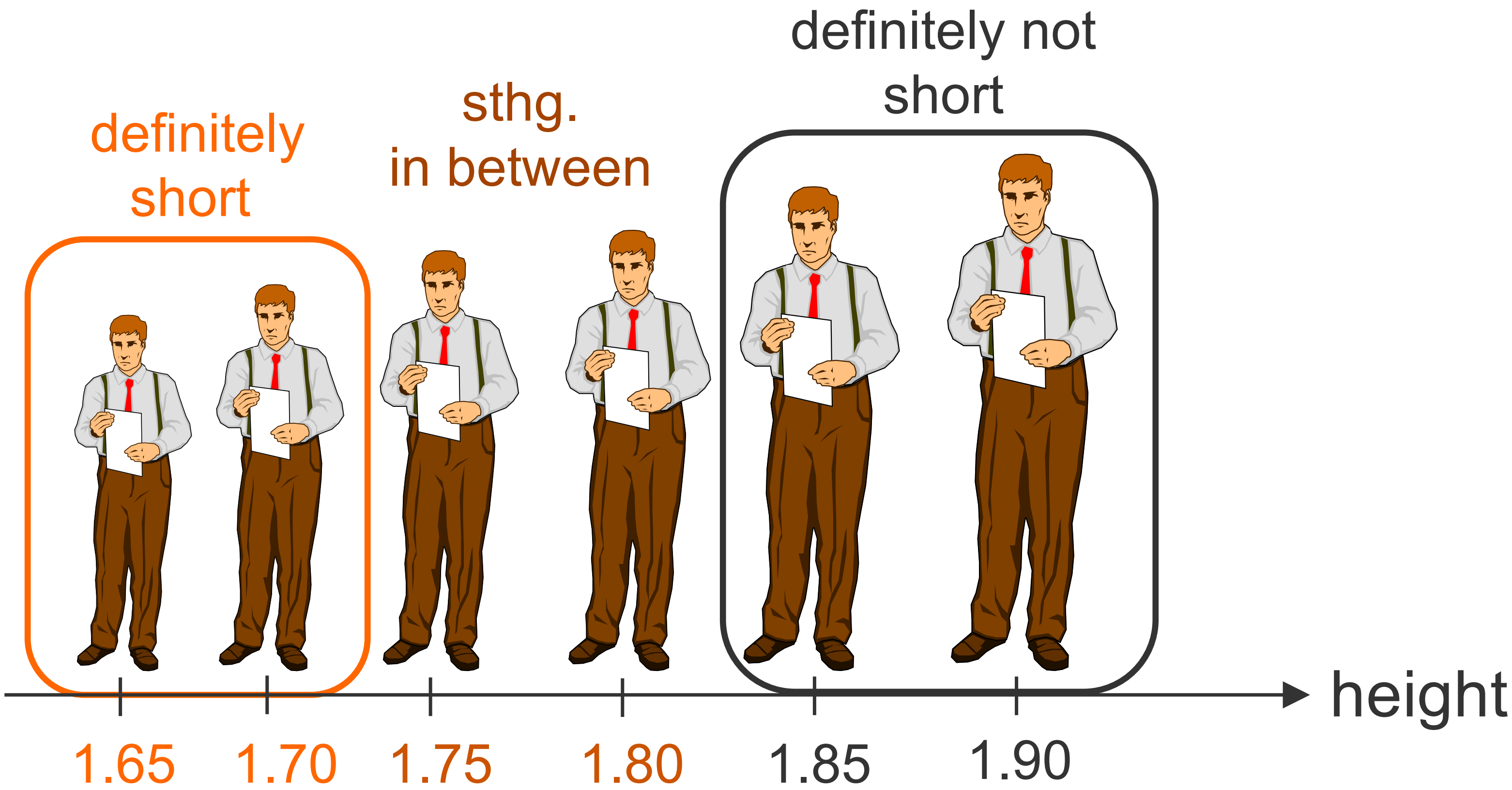
$$\mu_S(height) = \begin{cases} 1 \text{ if } hight \le 1.80 \\ 0 \text{ else} \end{cases}$$



height	*short?*
1.65	1
1.70	1
1.75	1
1.80	1
1.85	0
1.90	0

Representation by membership function

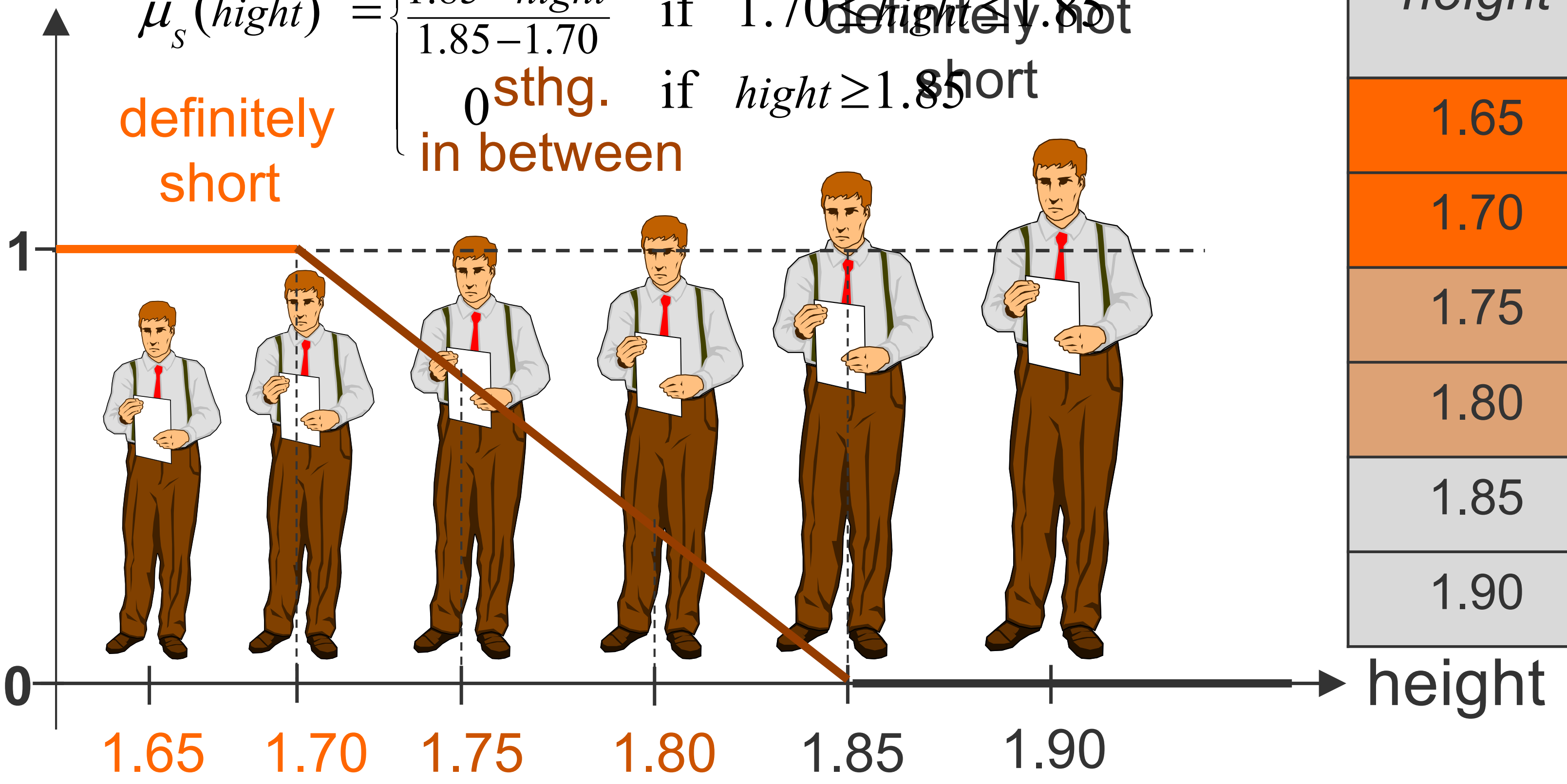
Fuzzy sets, e.g.: fuzzy set of short men



Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(hight) = \begin{cases} 1 & \text{if } hight \leq 1.70 \\ \frac{1.85 - hight}{1.85 - 1.70} & \text{if } 1.70 \leq hight \leq 1.85 \\ 0 & \text{if } hight \geq 1.85 \end{cases}$$

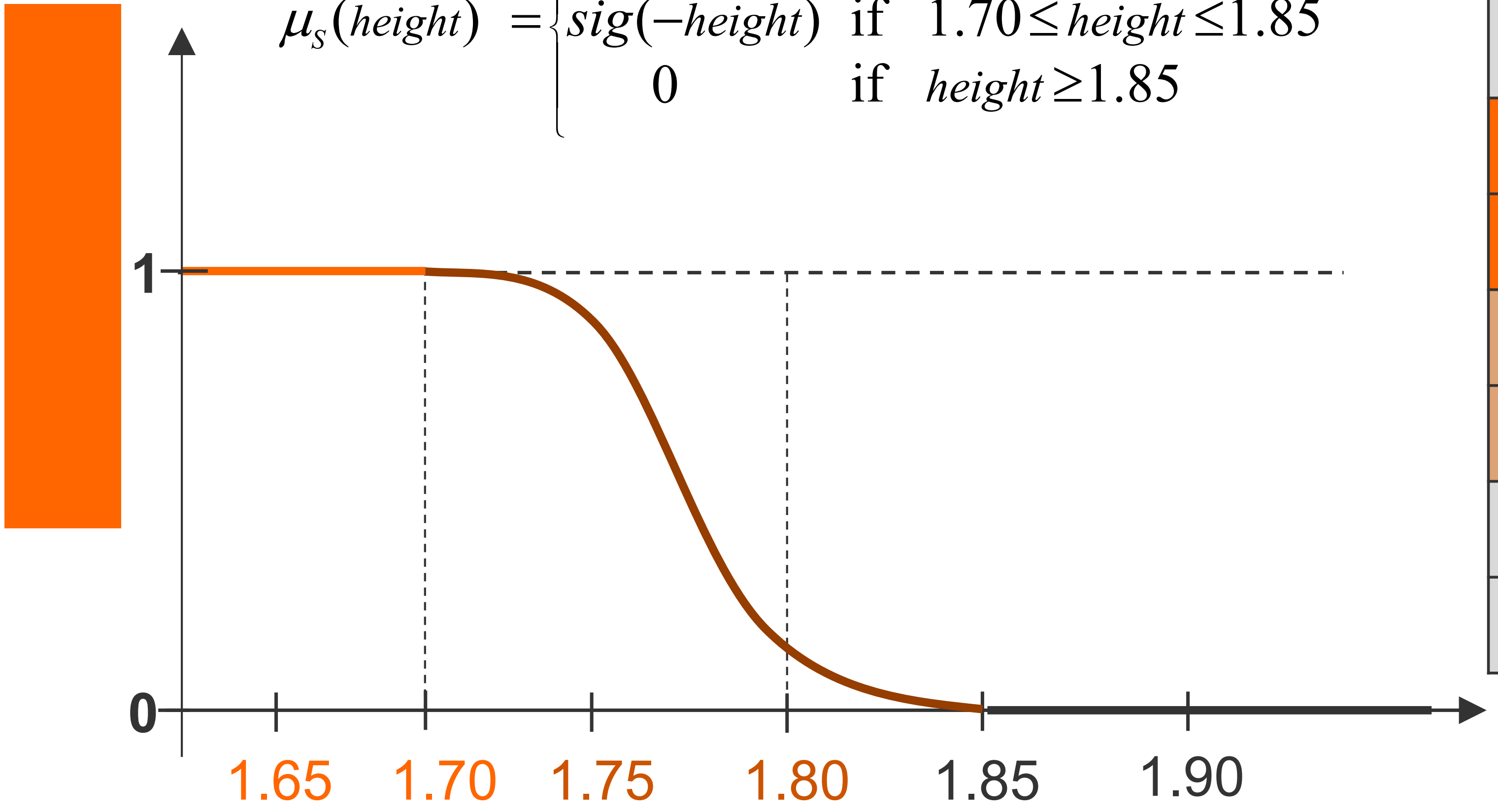


height	*short?*
1.65	1
1.70	1
1.75	2/3
1.80	1/3
1.85	0
1.90	0

Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(height) = \begin{cases} 1 & \text{if} \quad height \leq 1.70 \\ sig(-height) & \text{if} \quad 1.70 \leq height \leq 1.85 \\ 0 & \text{if} \quad height \geq 1.85 \end{cases}$$



height	*short?*
1.65	1
1.70	1
1.75	
1.80	
1.85	0
1.90	0

Exercise: Fuzzy Sets for Size of People

- Draw fuzzy sets for short, medium and tall men; use trapezoidal membership functions.
- Here are the restrictions:
  - Men below 1.60 are definitely short
  - Men taller than 175 are definitely not short
  - Men taller than 190 are definitely tall
  - Men smaller than 180 are not tall
  - Men between 170 and 185 are medium
  - Men below 165 are not medium
  - Men taller than 190 are not medium

FUZZY SET THEORY

Fuzzy Set Theory

Operations on Fuzzy Sets:

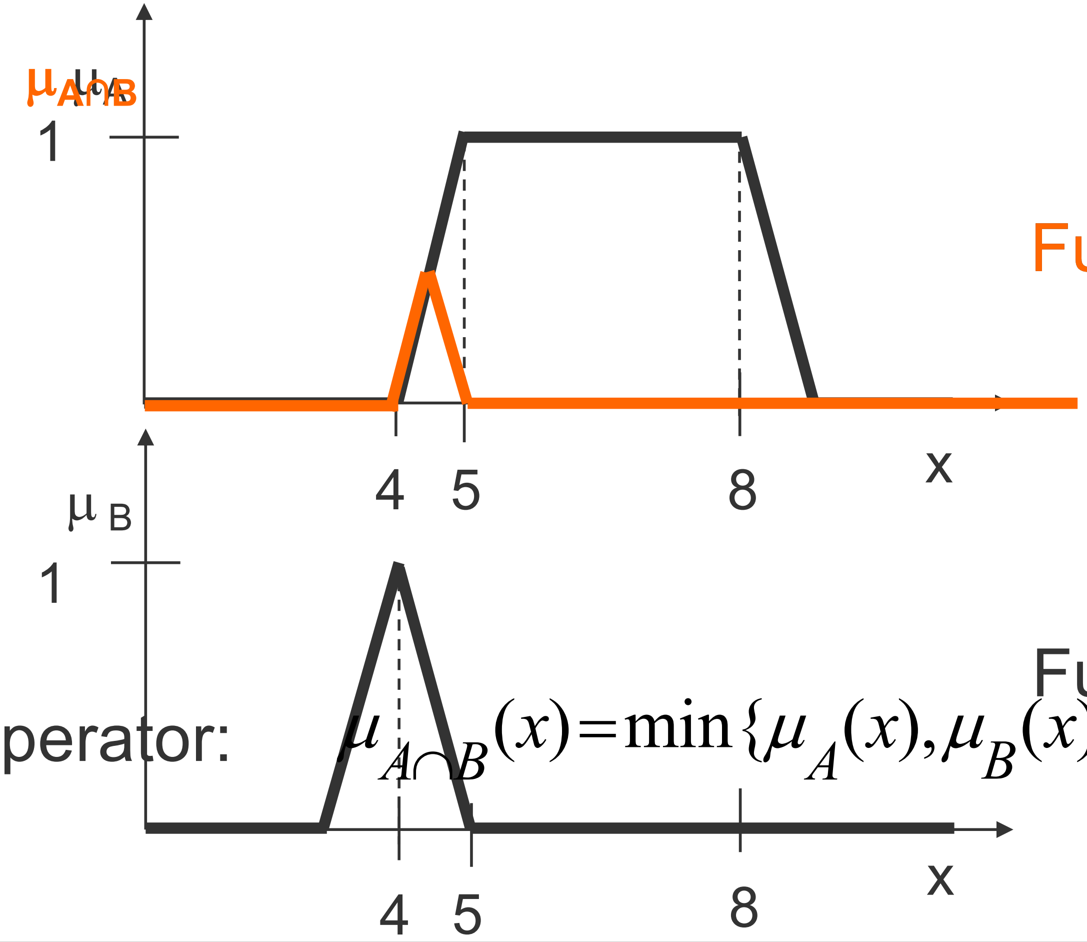
For Fuzzy Sets we can define operations

- ***intersection***,
- ***union***
- ***negation***

… analogue to classical sets.

Operations with Fuzzy Sets

Intersection:



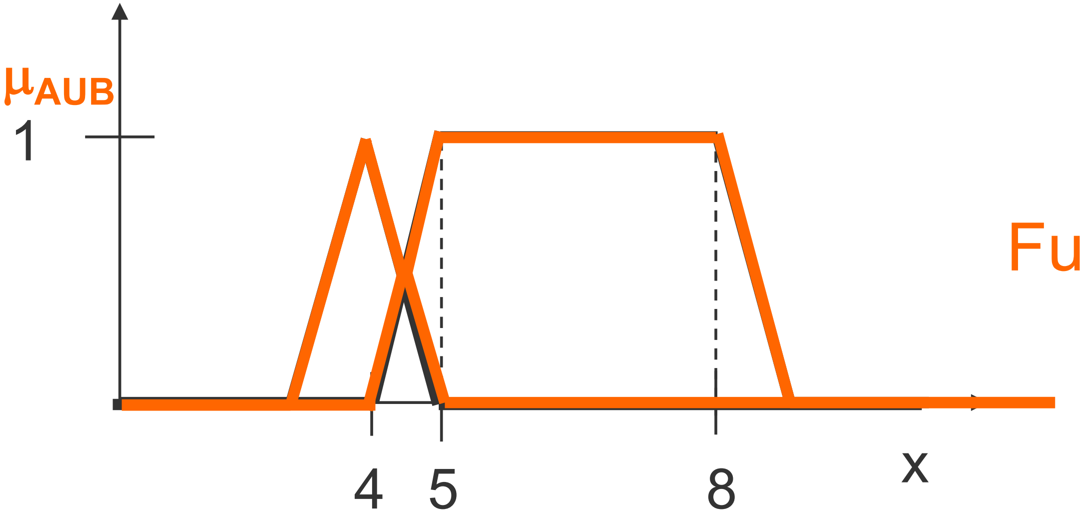
Fuzzy Set A∩B

Fuzzy Set B

Minimum Operator: $\mu_{A\cap B}(x)=\min\{\mu_A(x),\mu_B(x)\}$

Operations with Fuzzy Sets

Union:

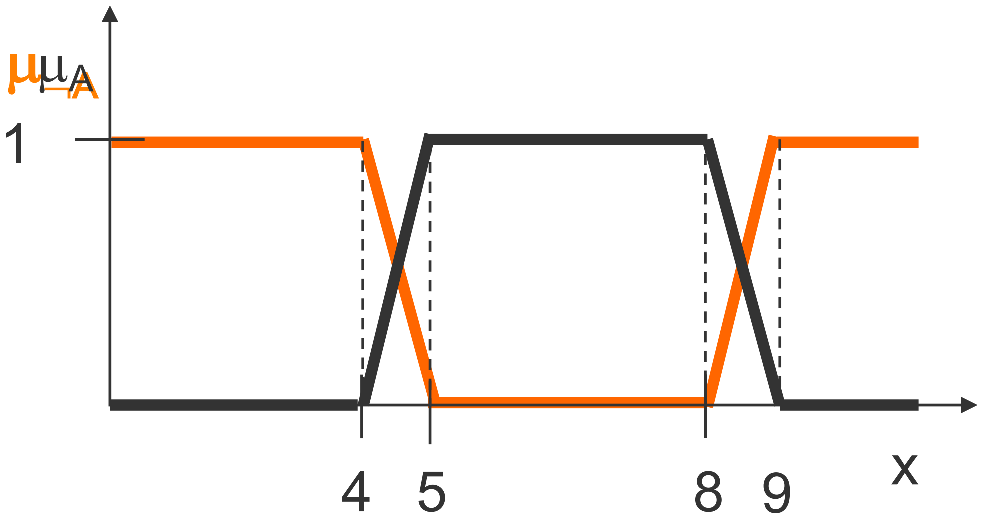


Fuzzy Set AUB

Maximum Operator: $\mu_{A\cup B}(x)=\max\{\mu_A(x),\mu_B(x)\}$

Operations with Fuzzy Sets

Negation:

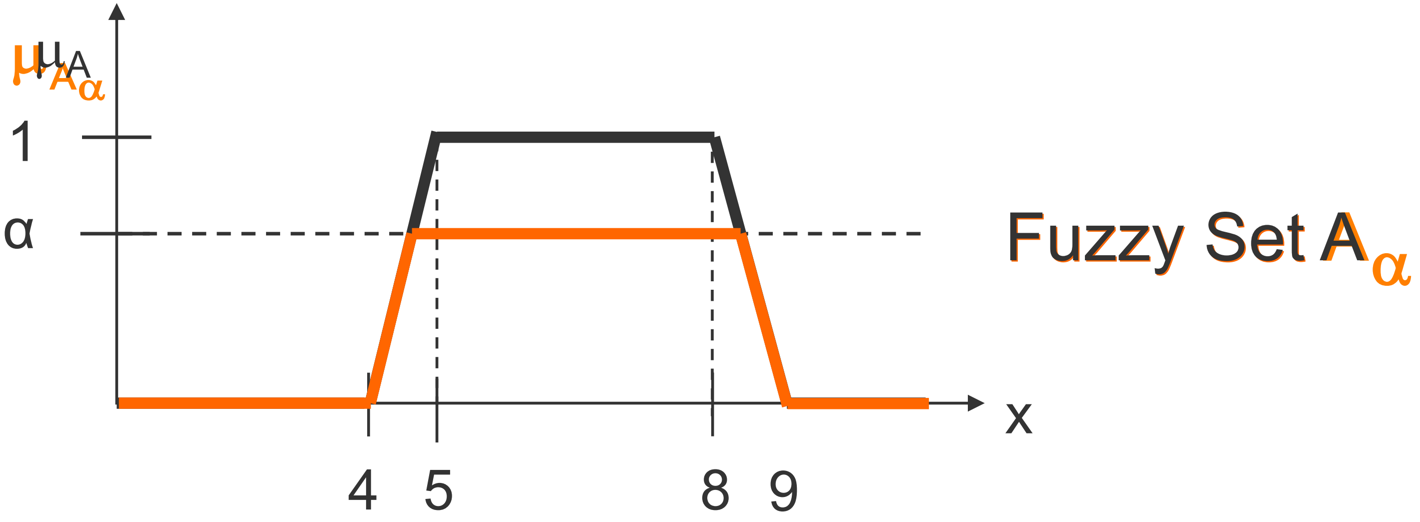


Fuzzy Set A ¬A

Complement Operator: $\mu_{\neg A}(x) = 1 - \mu_A(x)$

Operations with Fuzzy Sets

Alpha-cut:



α-Cut Operator:

$$\mu_{A_\alpha}(x) = \min\{\mu_A(x), \alpha\}$$

Exercise 2

FUZZY LOGIC

Fuzzy Logical Operators

- They modify or combine fuzzy logical statements.
  - E.g.: AND, OR, IMPLIES, NOT, …
- They are operations on membership degrees:
  - AND: minimum, $\mu_{A\wedge B}(x,y)=\min\{\mu_A(x),\mu_B(y)\}$
  - OR: maximum, $\mu_{A\vee B}(x,y)=\max\{\mu_A(x),\mu_B(y)\}$
  - NOT: complement $\mu_{\neg A}(x)=1-\mu_A(x)$
  - IMPLIES: minimum, $\mu_{A\rightarrow B}(x,y)=\min\{\mu_A(x),\mu_B(y)\}$

Mamdani Implication

- Note: There are serveral possibilities to define fuzzy logic operators! We use the above.

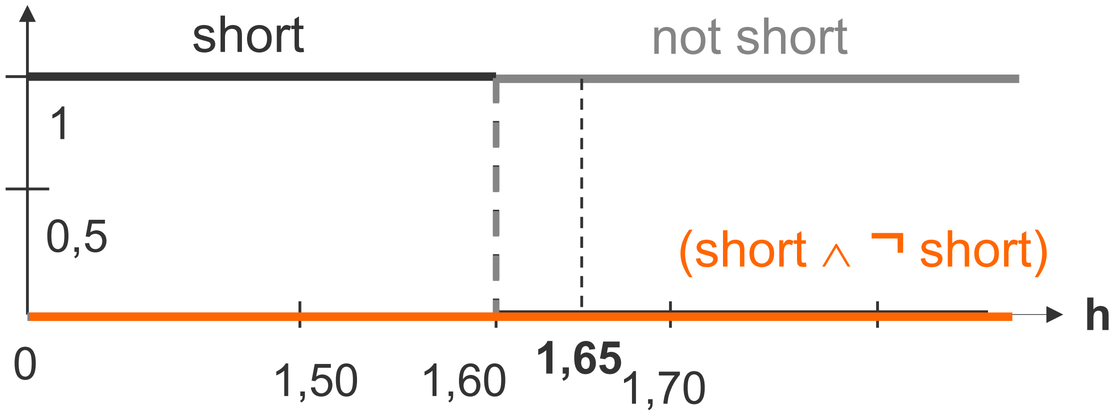
Fuzzy Logic „Paradox“

In classical logic, a statement and its negation cannot be true at the same time:

$$(s \wedge \neg s) = 0$$

„Tertium non datur“
(law of the excluded middle)

Example: Classical statement s=„Bob is short“,
where *short* is specified by the following crisp set:



If height(Bob)=1.65, then $(s \wedge \neg s) = \min\{0,1\}=0$.

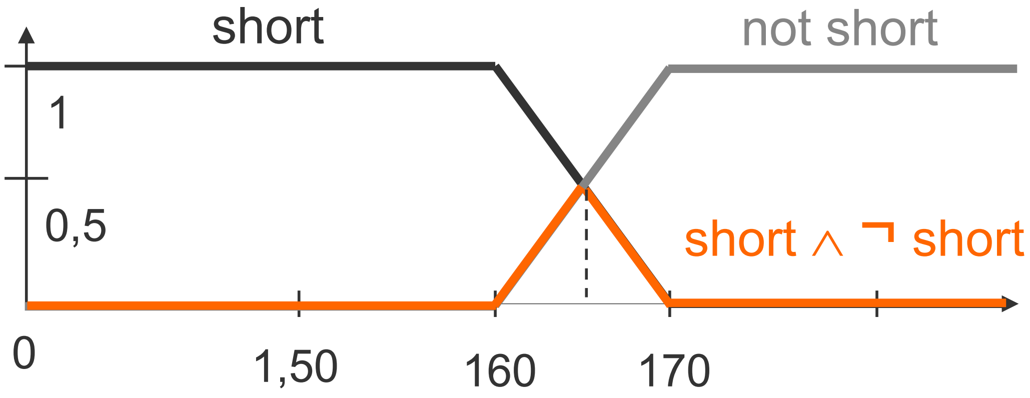
Fuzzy Logic „Paradox“

In fuzzy logic, a statement and ist negation can both be (partially) true at the same time:

$$(s \wedge \neg s) \neq 0 \qquad \text{for some } s$$

Example: Fuzzy statement s= „Bob is short“,

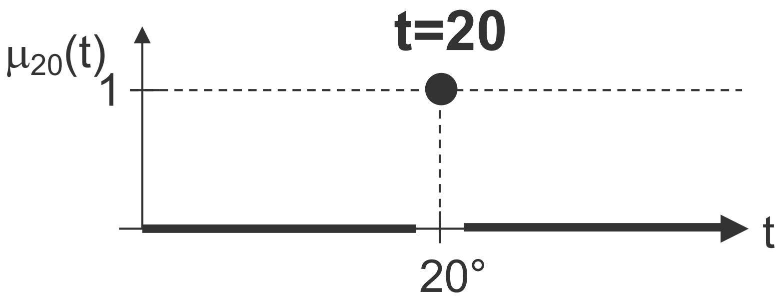
where short is specified by the following fuzzy set:



If height(Bob)=1.65, then $(s \wedge \neg s) = \min\{0.5,0.5\}=0.5$

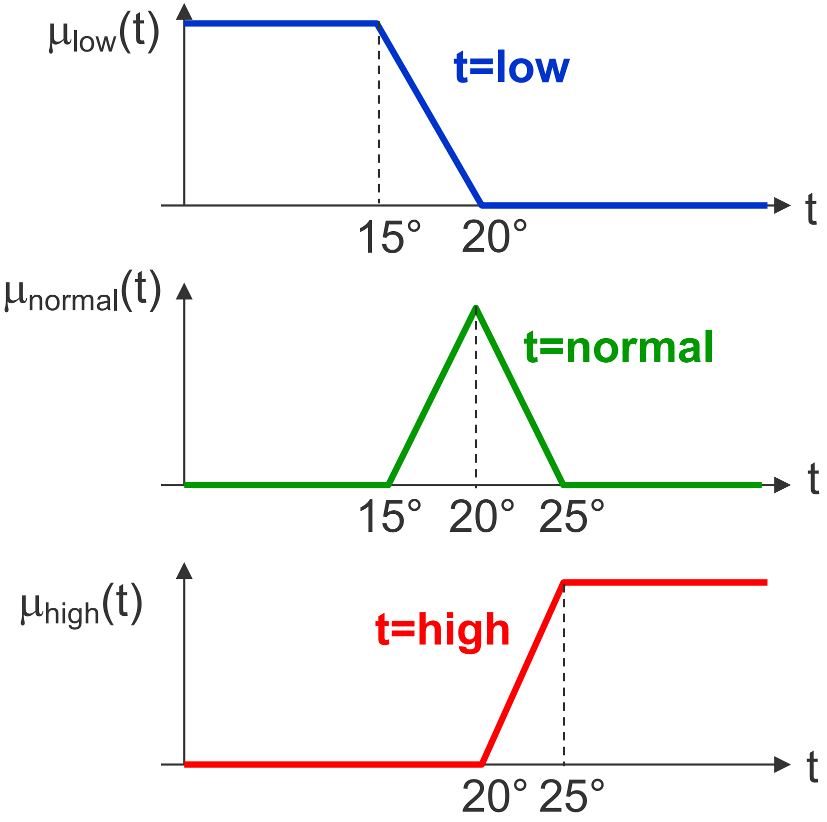
Classical vs. Linguistic Variables

Example: Classical variable «**temperature**» (t).
t takes exact values in the interval **[-50,50]**,e.g., t=20:

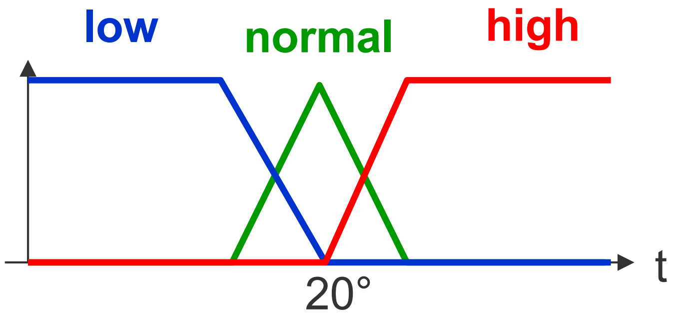


Classical vs. Linguistic Variables

Example: Linguistic variable «**temperature**» (t).
t takes the fuzzy values ***low, normal, high***,*e.g., t=low.*
Fuzzy values are defined as **Fuzzy Sets**:



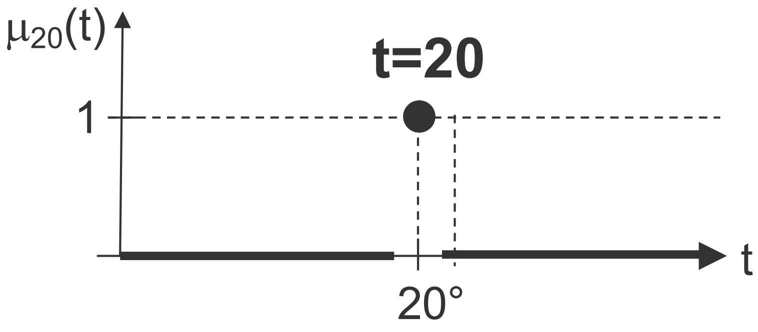
In one graphic:



Fuzzy Logical Statements

The possible truth values of an exact statement are:
1 (True) or 0 (False).

Example: Exact statement s=«The temperature is 20°C.»



«Temperature» is a classical variable (t).
Takes the value t=20.

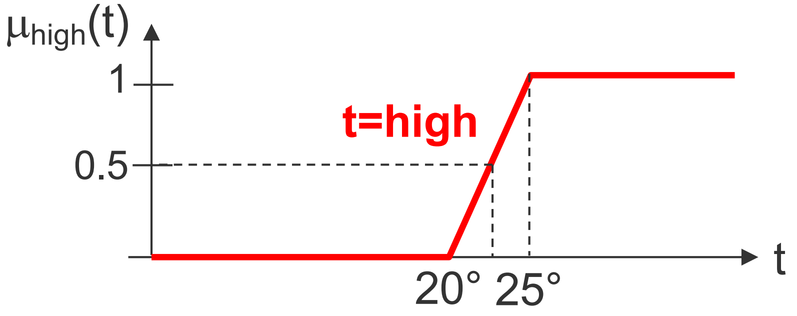
Assume the temperature is 22.5°C.

Then the truth value of s is 0.

Fuzzy Logical Statements

The possible truth values of a fuzzy statement are 1 (True), 0 (False), and every value in between.

Example: Fuzzy statement s=«The temperature is high.»



«Temperature» is a linguistic valriable (t). Takes the value t=high.

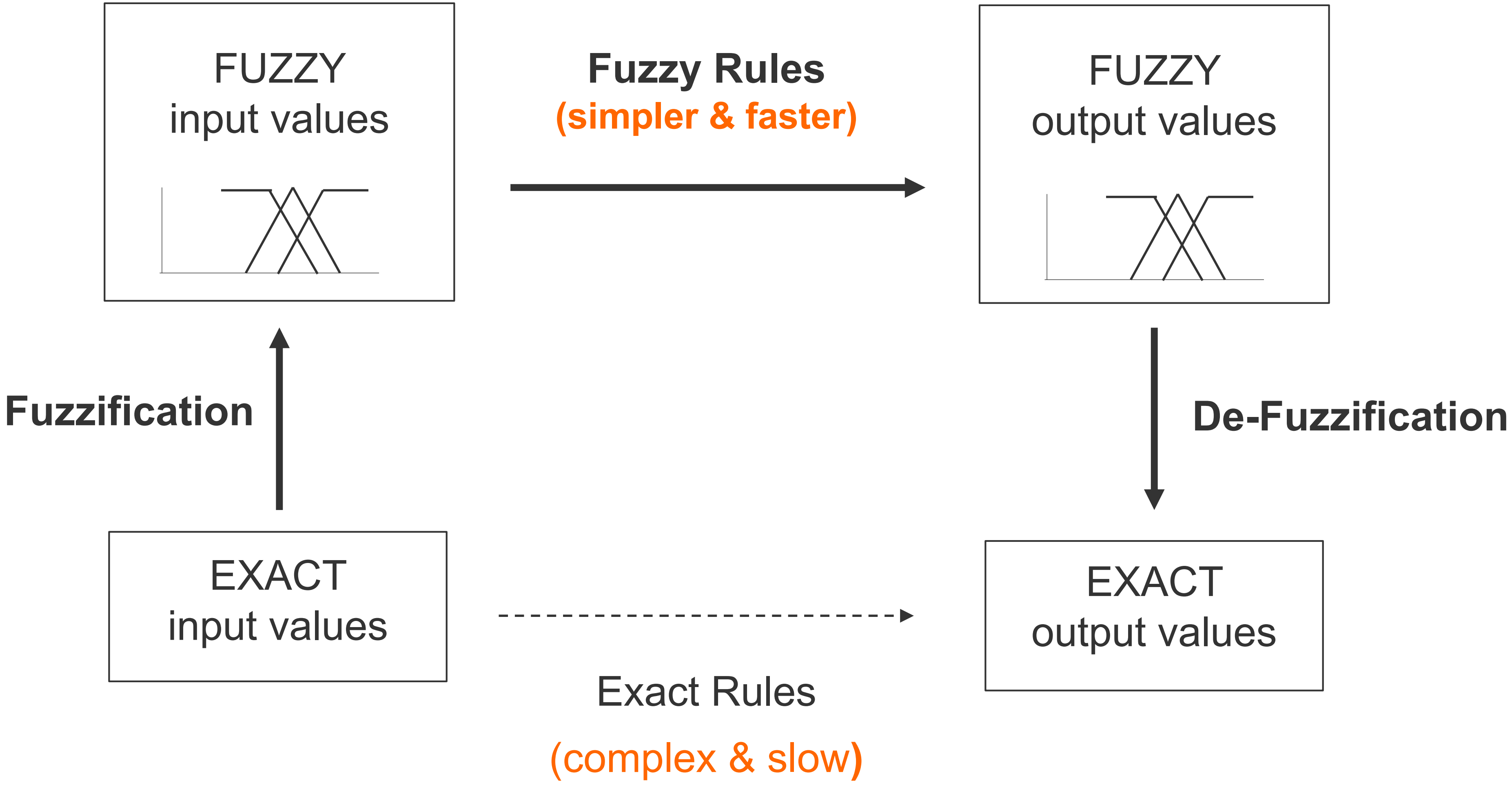
Assume the temperature is 22.5°C.

Then the truth value of s is 0.5.

The truth value of a fuzzy statement is also called truth degree. The truth degree indicates the degree of compatibility of the exact value 22.5°C with the fuzzy statements s.

FUZZY CONTROLLER

Designing a Fuzzy Controller (Procedure)



FUZZY
input values

Fuzzy Rules
(simpler & faster)

FUZZY
output values

Fuzzification

De-Fuzzification

EXACT
input values

EXACT
output values

Exact Rules

(complex & slow)

Input variables

Output variables

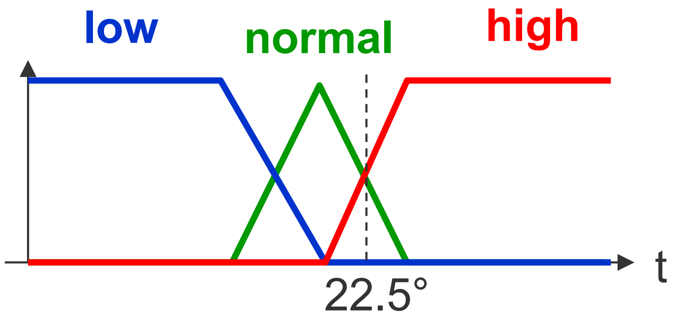
Fuzzification

- Transformation of exact variables to linguistic variables, and
- Transformation of exact values to fuzzy values (fuzzy sets).

Example: Fuzzification of variable «temperature»:

$$t \in [-50,50] \;\rightarrow\; t \in \{low,\ normal,\ high\}$$

$$t = 22.5°C \;\rightarrow\; \{\mu_{low}(t)=0,\ \mu_{normal}(t)=0.5,\ \mu_{high}(t)=0.5\}$$



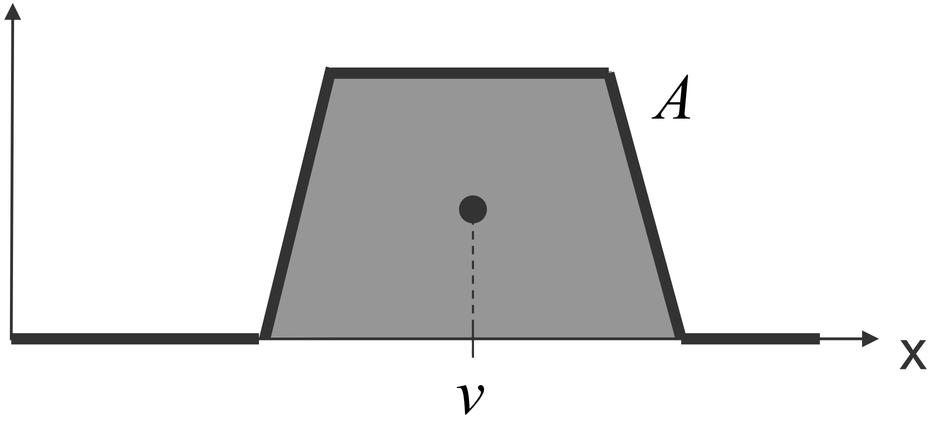
Defuzzification

= Transformation of a fuzzy set to an exact value (number).

Different possible methods, e.g.,

- Center of gravity method
- Maximum method
- Weighted average method

Example: Centre of gravity method (Sugeno 1985, most commonly used):



$$v = \frac{\int \mu_A(x) \cdot x\, dx}{\int \mu_A(x)\, dx}$$

Disadvantage: Computationally difficult for complex membership functions.

Fuzzy Logic Controller : „Car heating system”

- **Problem:** Car heating system
  - The heating systems of a car should keep the temperature constant.
  - The heating power that is necessary depends on the **temperature** and the **air humidity** in the car:
    - The *higher* the temperature, the *lower* must be the heating power.
    - The *lower* the temperature, the *higher* must be the heating power.
    - The humidity interacts with temperature.
  - Sensors show the current temperature and humidity.

Fuzzy Logic Controller „Car heating system”

Steps to build the fuzzy controller

1. Specify Input and Output variables
2. Fuzzification of variables and values
3. Define fuzzy rules
4. Choose defuzzification method

Fuzzy Logic Controller „Car heating system”

Step 1: Specify Input and Output variables

humidity

temperature

heating power

Input valiables

Output variables

Fuzzy Logic Controller „Car heating system”

Step 2: Fuzzification of variables and values:

- Transform exact variables in linguistic variables:
  - Humidity: {low, medium, high}
  - Temperature: {low, normal, high}
  - Heating power: {low, normal, increased, high}
- Specify the fuzzy values of the linguistic variables as fuzzy sets:
  - see next slide!

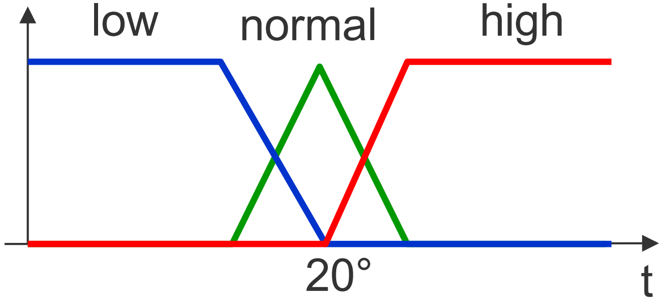
Fuzzy Logic Controller „Car heating system”

Step 2: Fuzzification of variables and values:

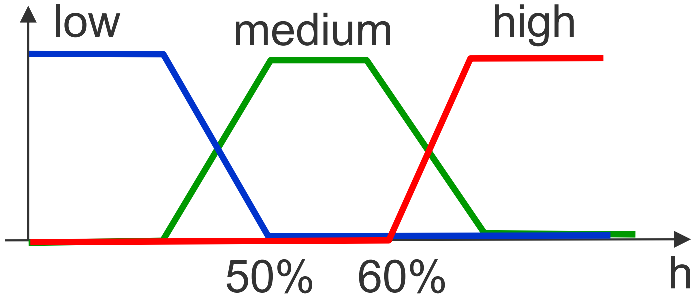
Input variables:

Fuzzy values:

Temperature



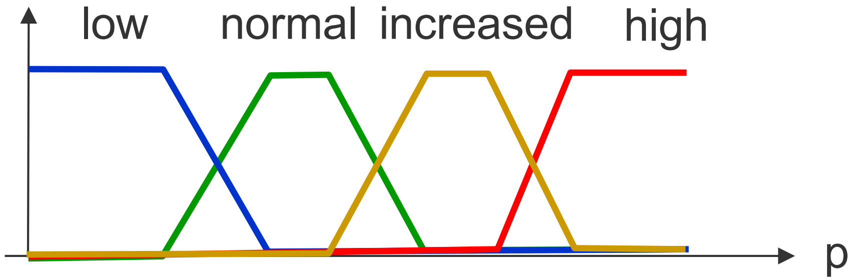
Humidity



Output variable:

Fuzzy values:

Heating Power



Fuzzy Logic Controller „Car heating system”

Step 3: Define fuzzy IF-THEN rules

- A fuzzy IF-THEN rule is NOT a logical implication, but can be thought of as a command.
- A set of Fuzzy IF-THEN rules maps linguistic variables to lingustic variabes (fuzzy function).
- Fuzzy IF-THEN rule describes the control of the system. They are similar to the experiences of an expert, who would formulate their knowledge in natural language terms.

Fuzzy Logic Controller „Car heating system”

Step 3: Define fuzzy IF-THEN rules

- Rule 1:
  - IF Temperature = *low*
    THEN heating power is *increased*
- Rule 2:
  - IF Temperature = *normal* AND humidity = *low*
    THEN heating power is *normal*
- Rule 3:
  - IF Temperature = *normal* AND humidity = *high*
    THEN heating power is *high*
- Rule 4:
  - IF Temperature = *high*
    THEN heating power is *low*

Fuzzy Logic Controller „Car heating system”

Step 3: Define fuzzy IF-THEN rules … as decision table

Humidity

Temperature

AND	**low**	**medium**	**high**
low	increased	increased	increased
normal	normal		high
high	low	low	low

White fields contain irrelevant cases

Fuzzy Logic Controller „Car heating system"

Step 4: Choose defuzzification method

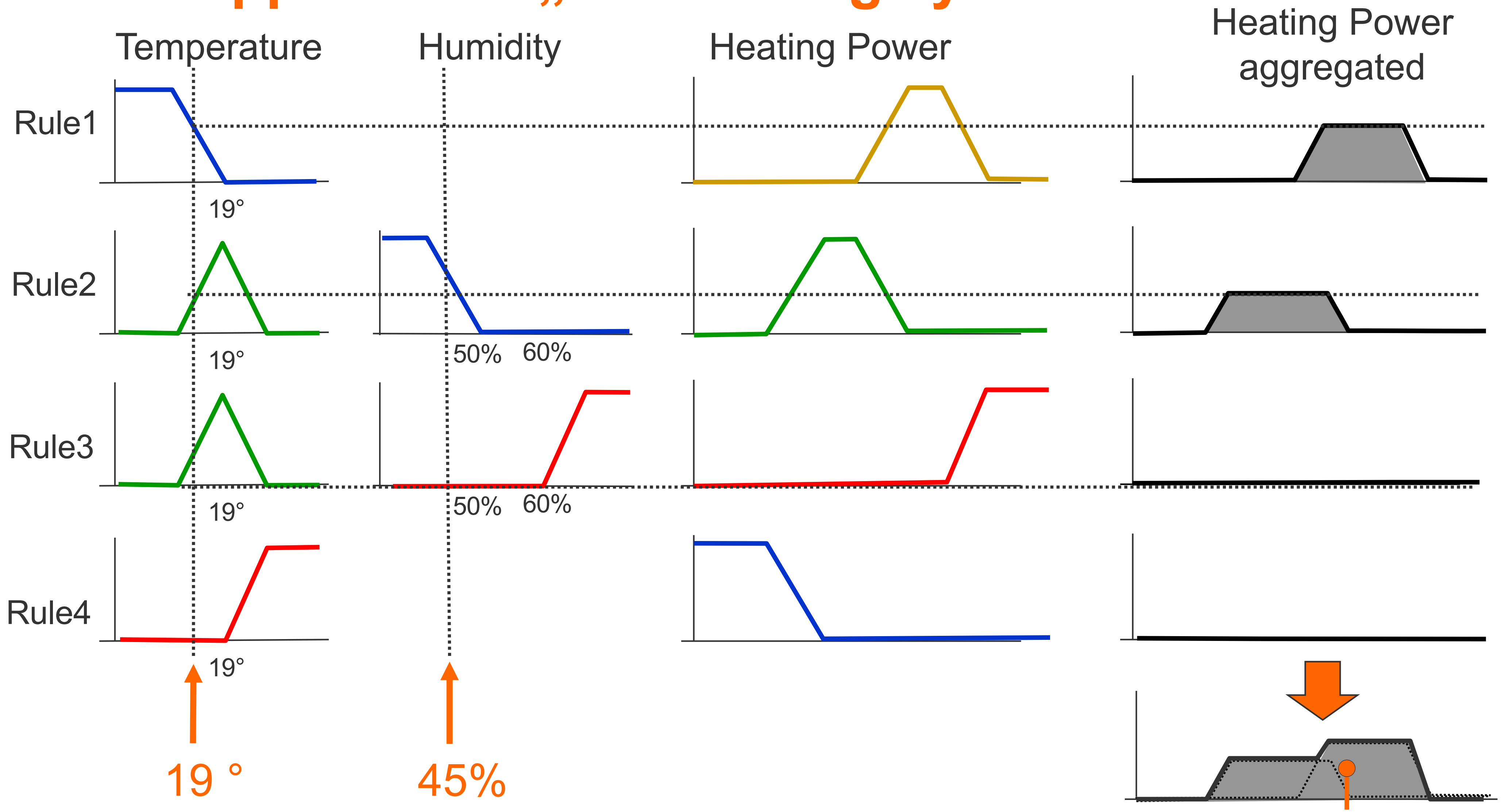
- The output of the fuzzy IF-THEN rules is a fuzzy value, i.e., a fuzzy set.
- The fuzzy output value must be mapped to an exact value in order to control the machine (the heating power engine).
- E.g., choose the centroid method (commonly used for fuzzy control systems).

Rule Application: „Car heating system”

Rule Application is performed in four steps:

1. Evaluate Antecedents:
   - For an exact input value, determine to which degree each antecedent is satisfied
   - Combine the degrees using the logical operators (AND in our example)
2. Evaluate Consequents:
   - The degree to wich an input variables A_i is satisfied determines the degree to which the corresponding output variable B_i holds (because IF-THEN rules are fuzzy functions). The result is the alpha cut of the output variable.
3. Aggregate Consequents:
   - Each rule gives one fuzzy set as a fuzzy output. Since all rules are valid, the fuzzy outputs may overlap (law of the excluded middle does not hold in general!). Combine them by OR to obtain a single fuzzy output value («aggregated output»).
4. Defuzzify Aggregated Output

Rule Application: „Car heating system"



Exact Sensor Input

Exact Sensor Output: v

Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

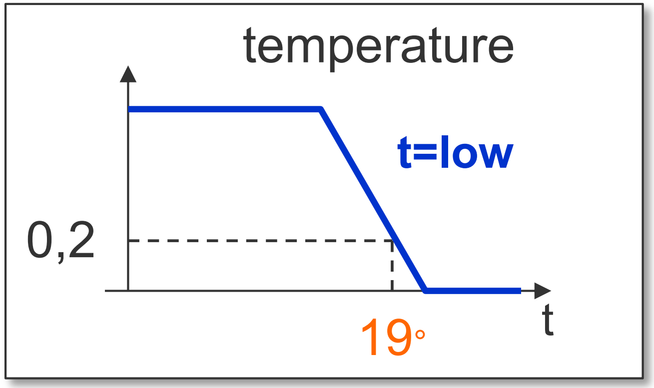
Assume the sensors have measured the following exact input values:

Temperature: t=19°
Humidity: h=45%

Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

Rule 1: IF temperature = *low*
THEN heating power is *increased*

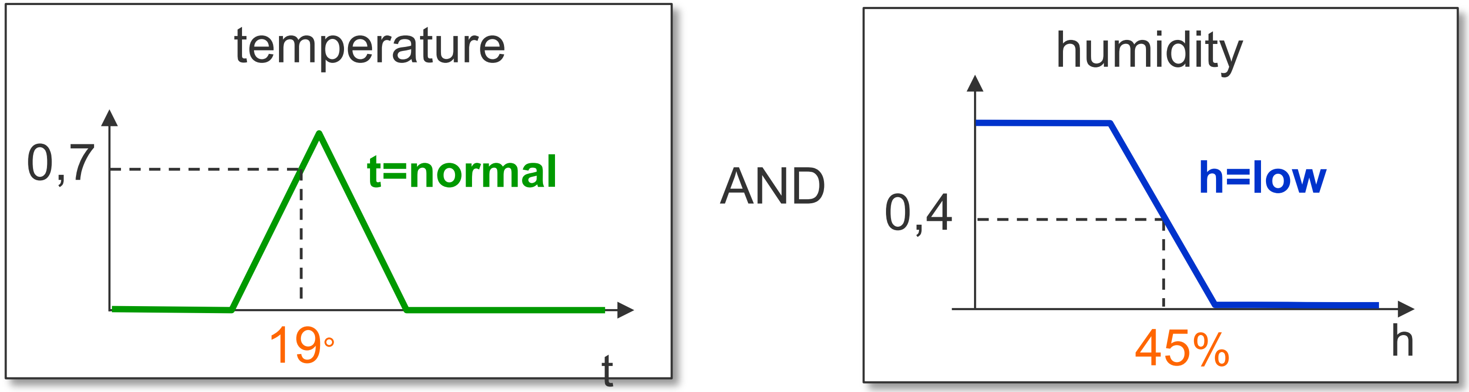


$$\mu_{t=low}(19°)=0.2$$

Rule Application: „Car heating system"

Step 1: Evaluate Antecedents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*



$$\mu_{t=normal}(19°) = 0.7$$

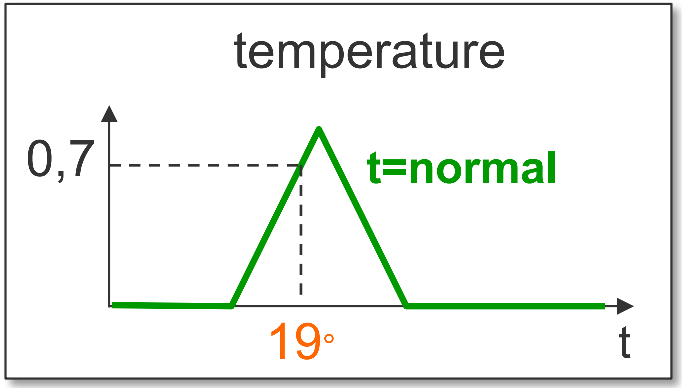
$$\mu_{h=low}(45\%) = 0.4$$

Min-Operator for AND: $$\mu_{t=normal \wedge h=low}(19°, 45\%) = \min\{0.7,\ 0.4\} = 0.4$$

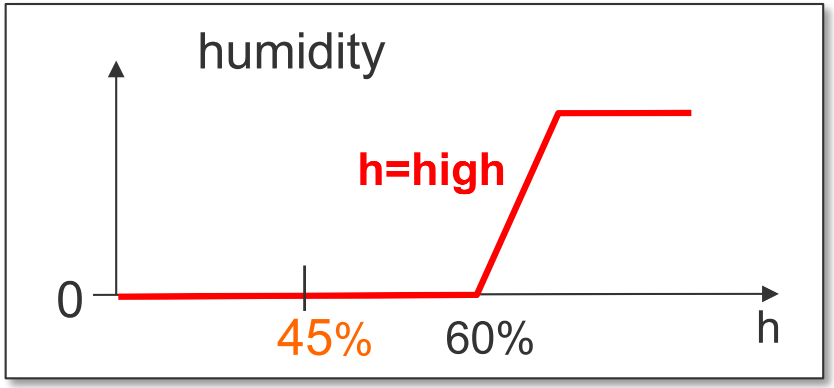
Rule Application: „Car heating system"

Step 1: Evaluate Antecedents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*



AND



$$\mu_{t=normal}(19°)=0.7$$

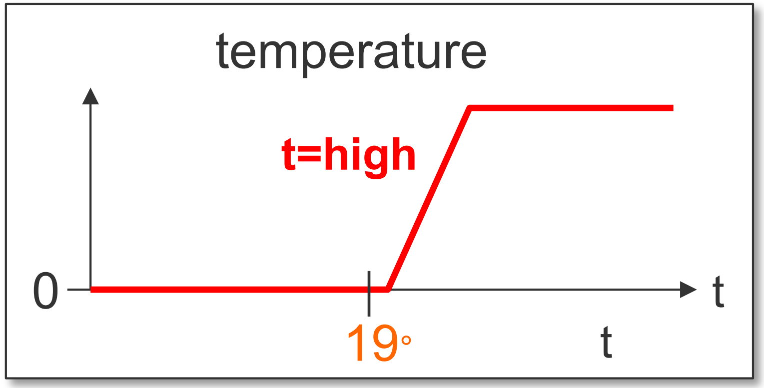
$$\mu_{h=high}(45\%)=0$$

Min-Operator for AND: $$\mu_{t=normal \wedge h=high}(19°,45\%)=\min\{0.7,\ 0\}=0$$

Rule Application: „Car heating system”

Step 1: Evaluate Antecedents

Rule 4: IF Temperature = *high*
THEN heating power is *low*

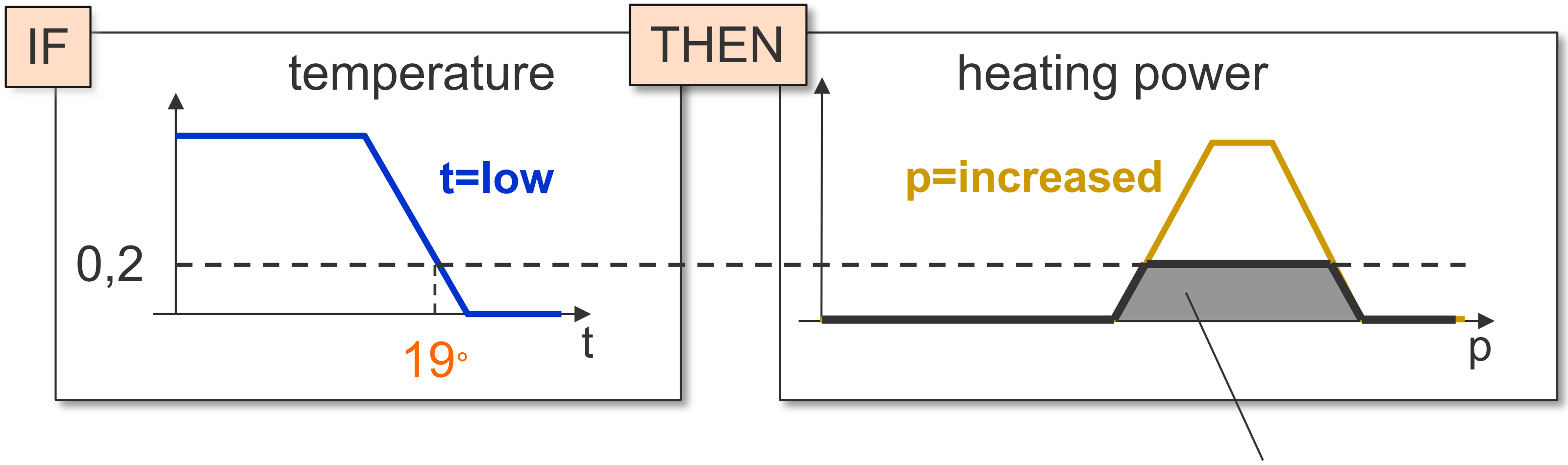


$$\mu_{t=high}(19°)=0$$

Rule Application: „Car heating system"

Step 2: Evaluate Consequents

Rule 1: IF temperature = *low*
THEN heating power is *increased*



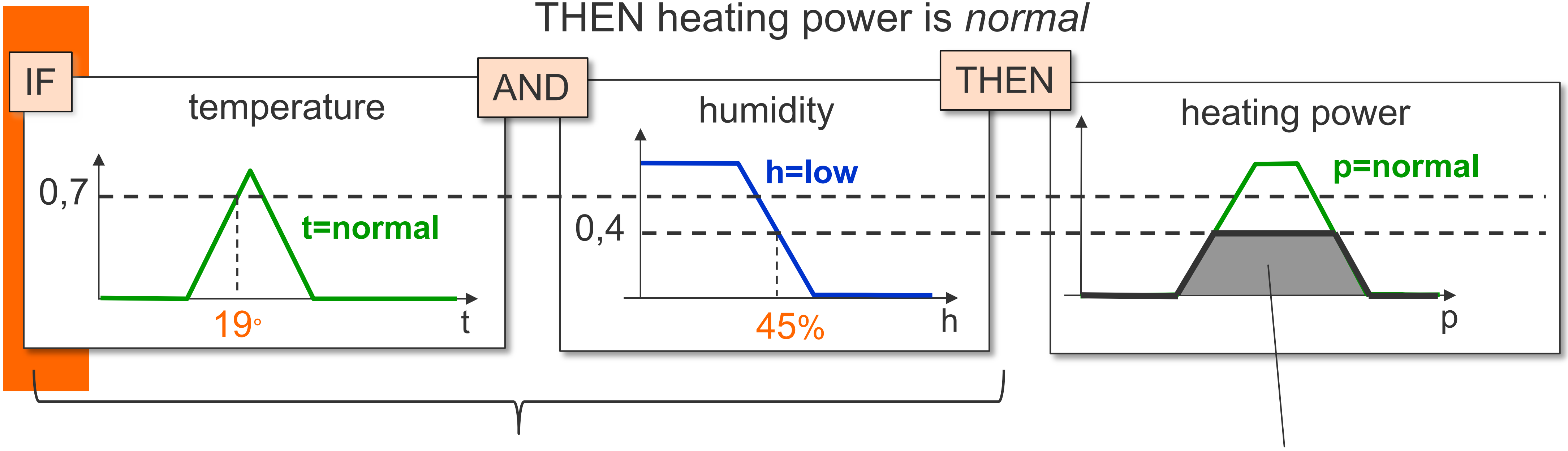
Output fuzzy set: $(\mu_{p=increased})_{0.2}$

(α-cut of $\mu_{p=increased}$ with $\alpha = 0.2$.)

Rule Application: „Car heating system”

Step 2: Evaluate Consequents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*



Min-Operator for AND:

$$\mu_{t=normal \wedge h=low}(19^{\circ},45\%)=0.4$$

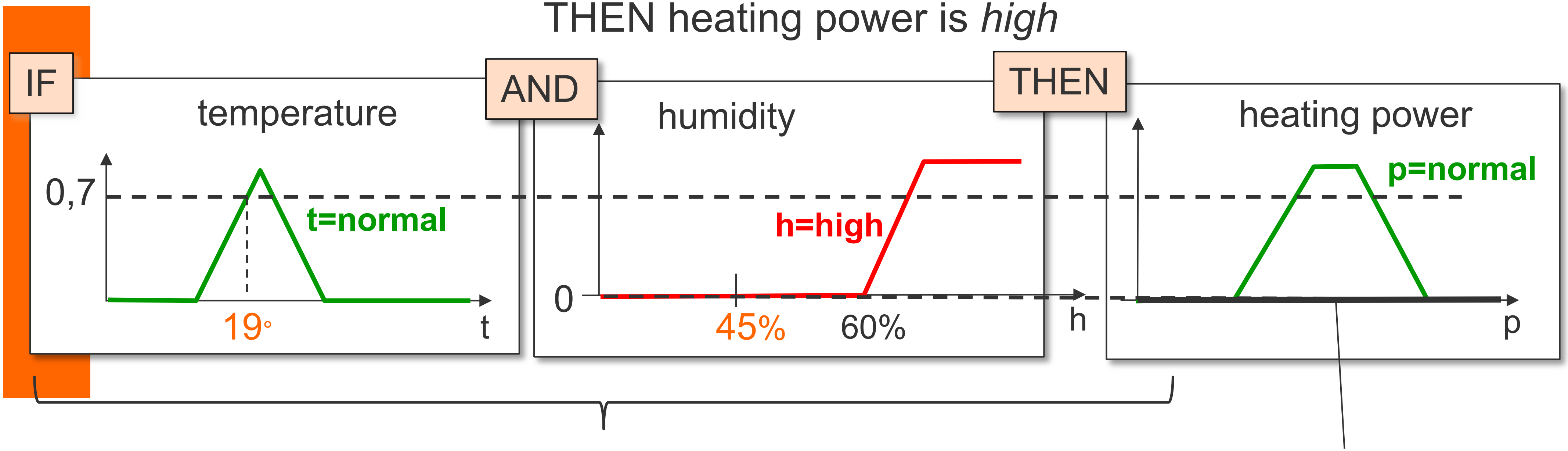
Output fuzzy set:

$$(\mu_{p=normal})_{0.4}$$

Rule Application: „Car heating system”

Step 2: Evaluate Consequents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*



Min-Operator for AND:

$$\mu_{t=normal \wedge h=high}(19°,45\%)=0$$

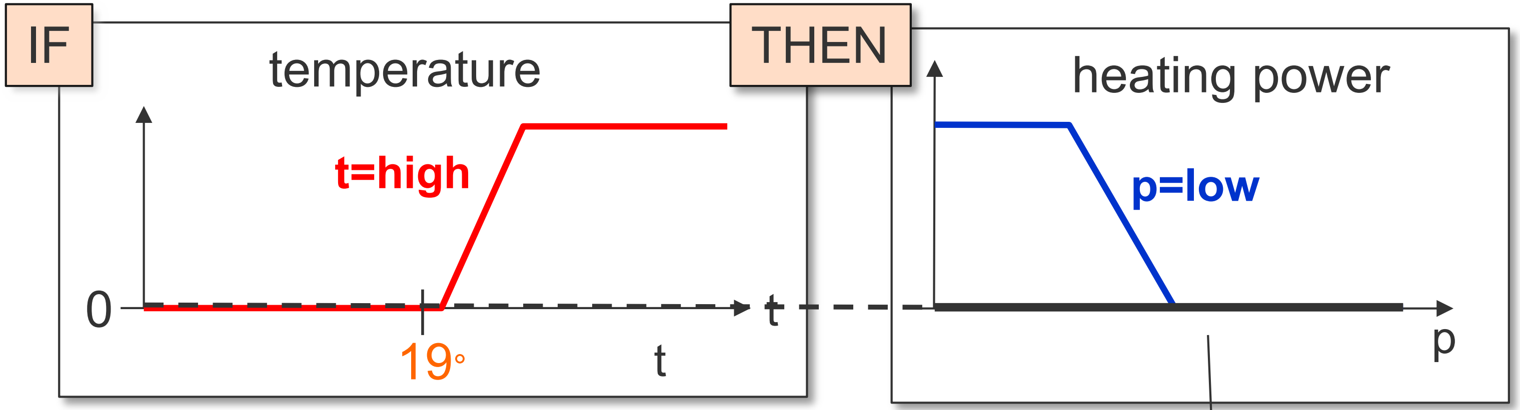
Output fuzzy set:

$$(\mu_{p=normal})_0 \equiv 0$$

Rule Application: „Car heating system”

Step 2: Evaluate Consequents

Rule 4: IF Temperature = *high*
THEN heating power is *low*



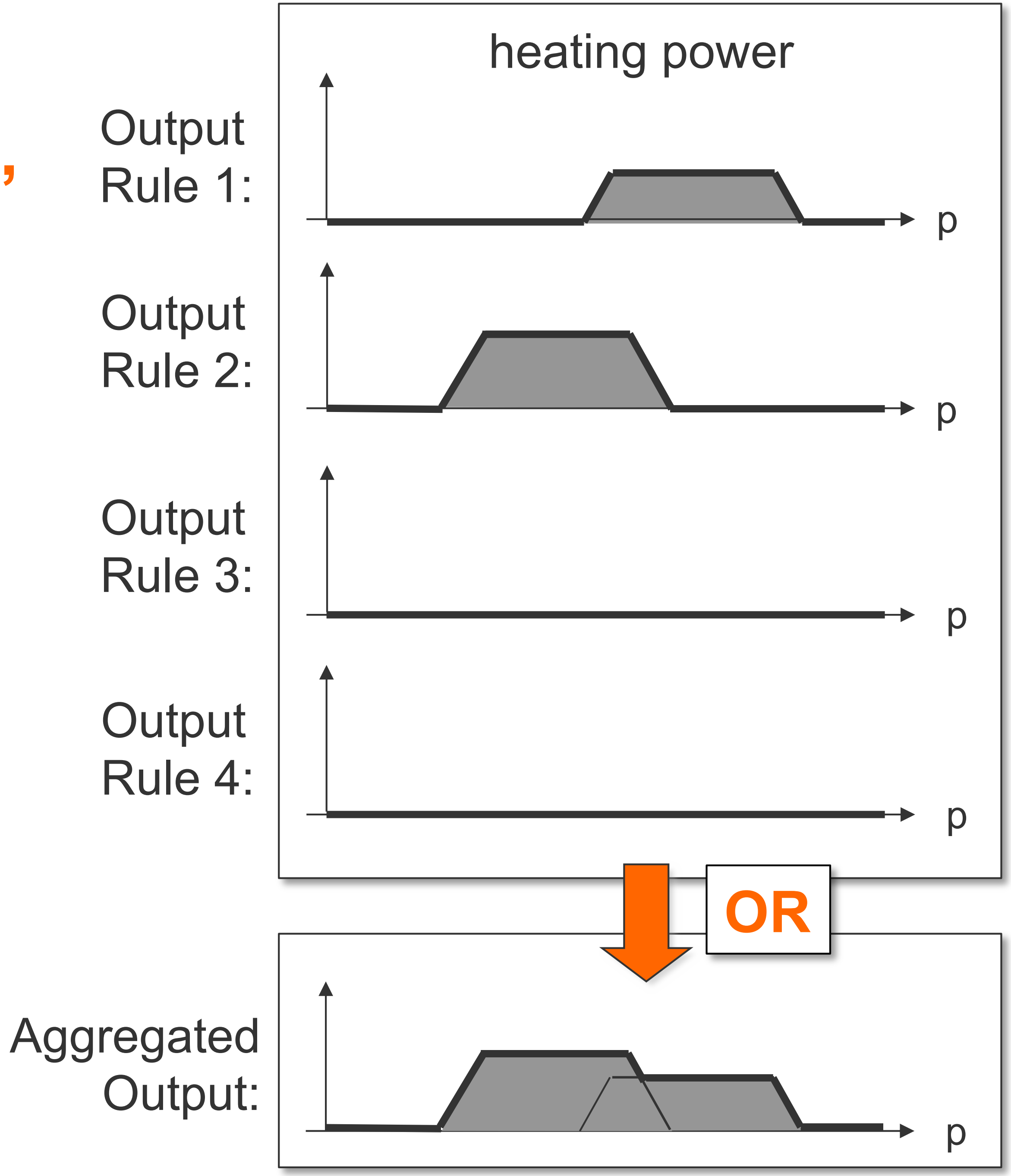
$$\mu_{t=high}(19°)=0$$

Output fuzzy set:

$$(\mu_{p=normal})_0 \equiv 0$$

Rule Application: „Car heating system"

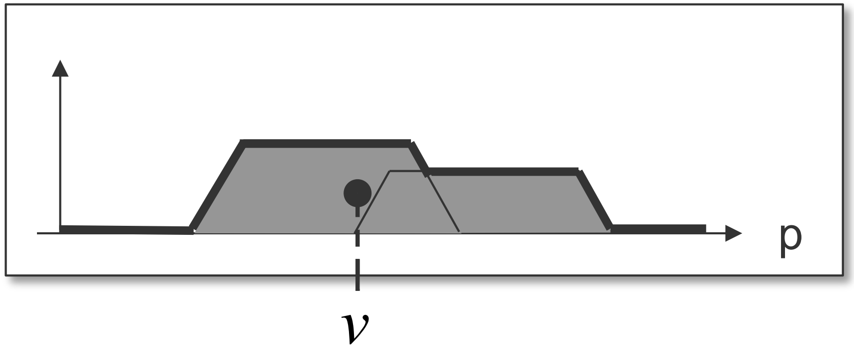
Step 3: Aggregate
Evaluated
Consequents:



Rule Application: „Car heating system”

Step 4: Defuzzify aggregated output

Center of gravity method:



$$v = \frac{\int \mu_A(x) \cdot x \, dx}{\int \mu_A(x) \, dx}$$

Rule Application: „Car heating system”

Main difference to exact reasoning:

Several rules can be active at the same time!
(Usually with different strengths.)