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# Vague Knowledge: Fuzzy Logic

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(with some slides from Holger Wache and Gwendolin Wilke)

# Repetition: Reliability of Knowledge

- Exact knowledge:
  - „It is raining."
- Uncertain knowledge:
  - „I believe it will not rain tomorrow.“
- Incomplete knowledge:
  - „The temperature ist between 10 and 15 degree Celsius“
  - "It will rain between 2 and 5 mm tomorrow"
- Vague knowledge (interpretation-dependent knowledge):
  - „The weather is good."

# Application

- Definition of rules with fuzzy values
- Example:
  - Fuzzy-Logic Controller for a heating controller

    IF Temperature = normal AND humidity = high
    THEN heating power = high

  - Product recommendation

    IF requirement = normal AND price = low
    THEN product = standard

# FUZZY SETS

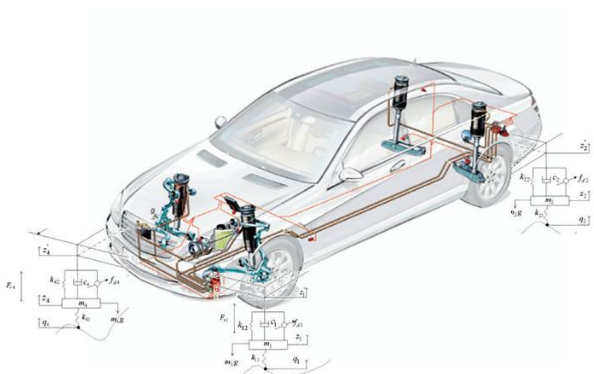
# Applications of Fuzzy Logic


- Fuzzy Systems became well-known as control systems (Washing machine, …)
- Defining rules with vague knowledge

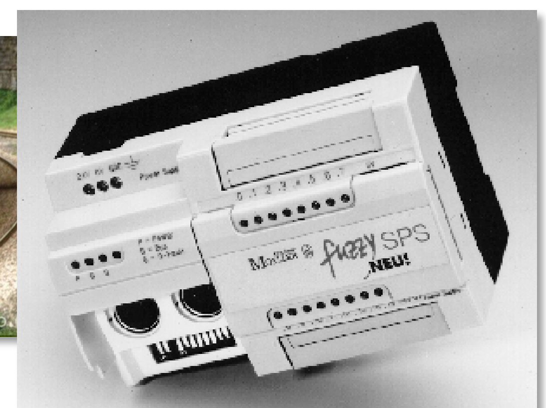
  IF Temperature = normal AND humidity = high
  THEN heating power = high


Washing Machine

- Other application areas: Diagnosis, Language understanding, …



Active Suspension Control System



# Inventor of Fuzzy Logic

Lotfi Zadeh 2010

Lotfi Zadeh 1945

# Classical vs. Fuzzy Sets

- Bald Men Paradox:
  - Would you describe a man with 1 hair on his head as bald? YES.
  - Would you describe a man with 2 hairs on his head as bald? YES.
  - Would you describe a man with 3 hairs on his head as bald? YES.

    ...........

  - Would you describe a man with 10000 hairs on his head as bald? NO.

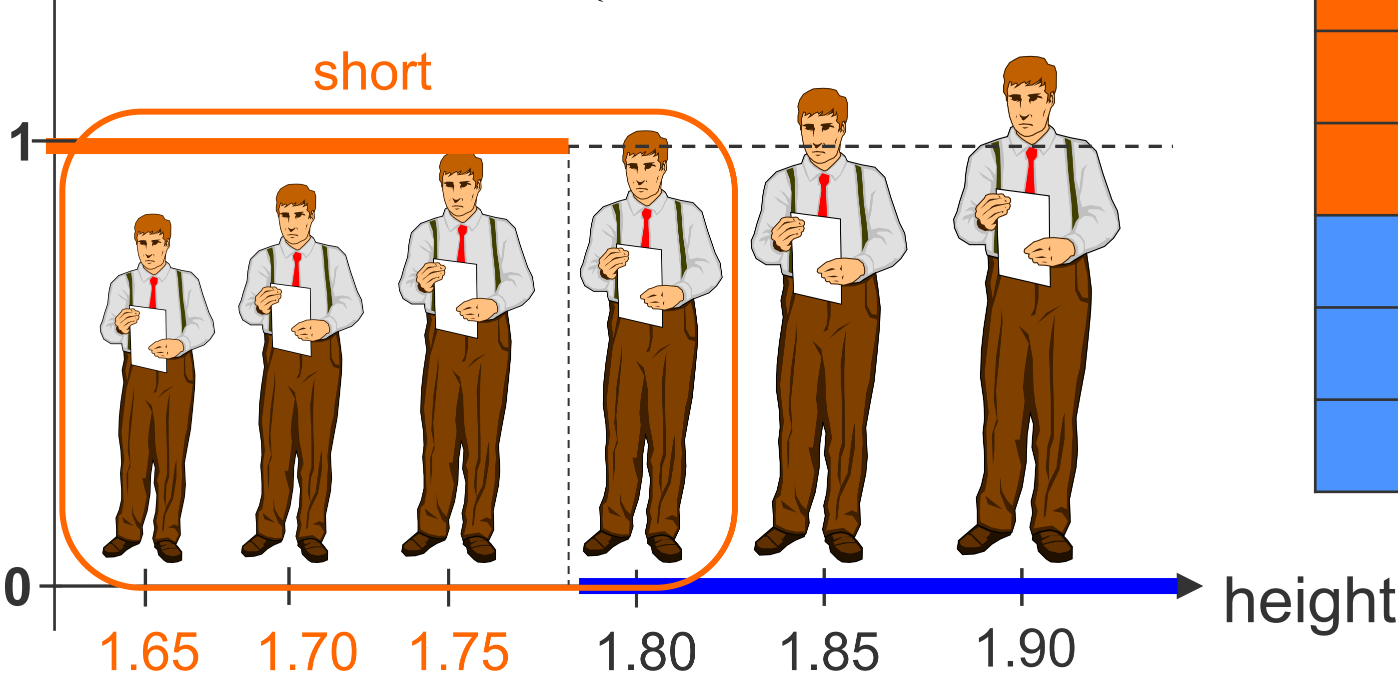
Where to draw the line?

# Who is short and who is tall? And who is medium?

# Representation by membership function

Classical sets, e.g.: set of short men S= {m | height(m) ≤ 1.80}

$$\mu_S(height) = \begin{cases} 1 & \text{if } hight \leq 1.80 \\ 0 & \text{else} \end{cases}$$




| *height* | *short?* |
|---|---|
| 1.65 | 1 |
| 1.70 | 1 |
| 1.75 | 1 |
| 1.80 | 0 |
| 1.85 | 0 |
| 1.90 | 0 |

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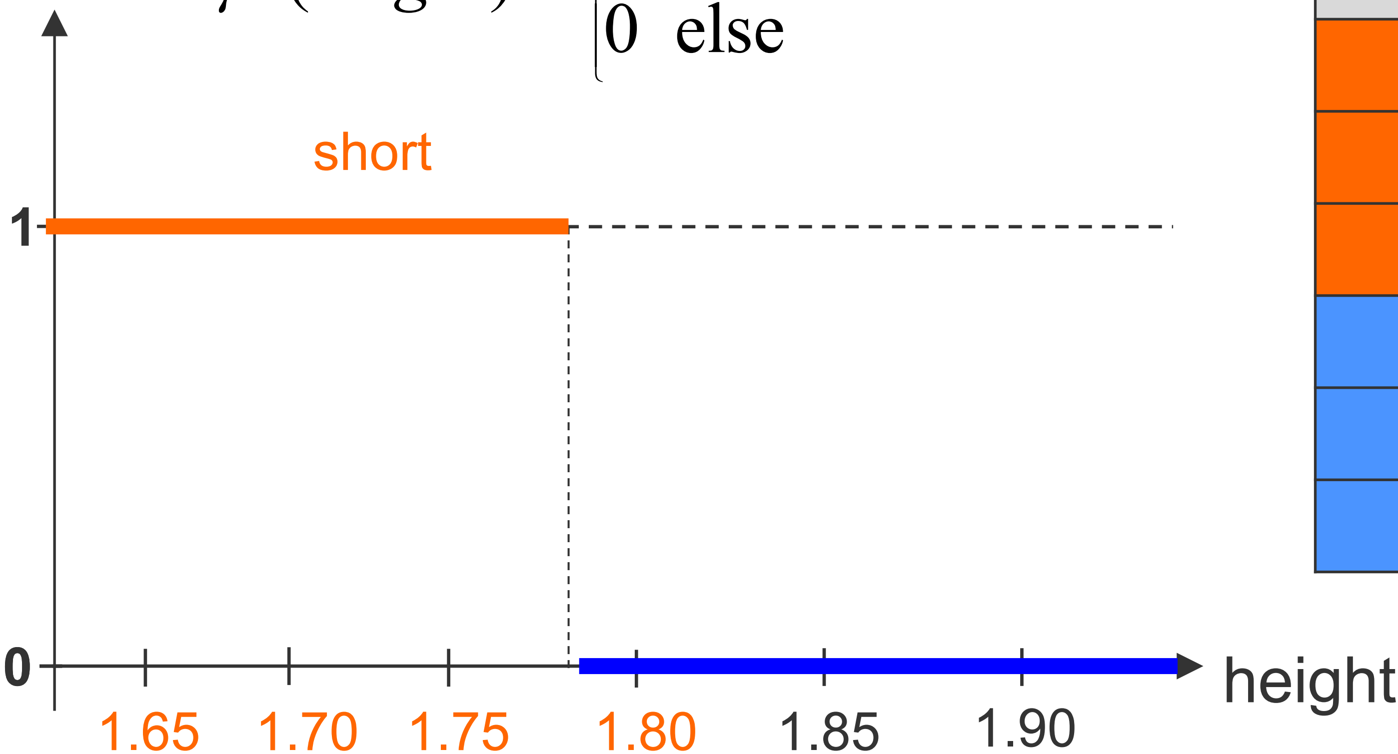
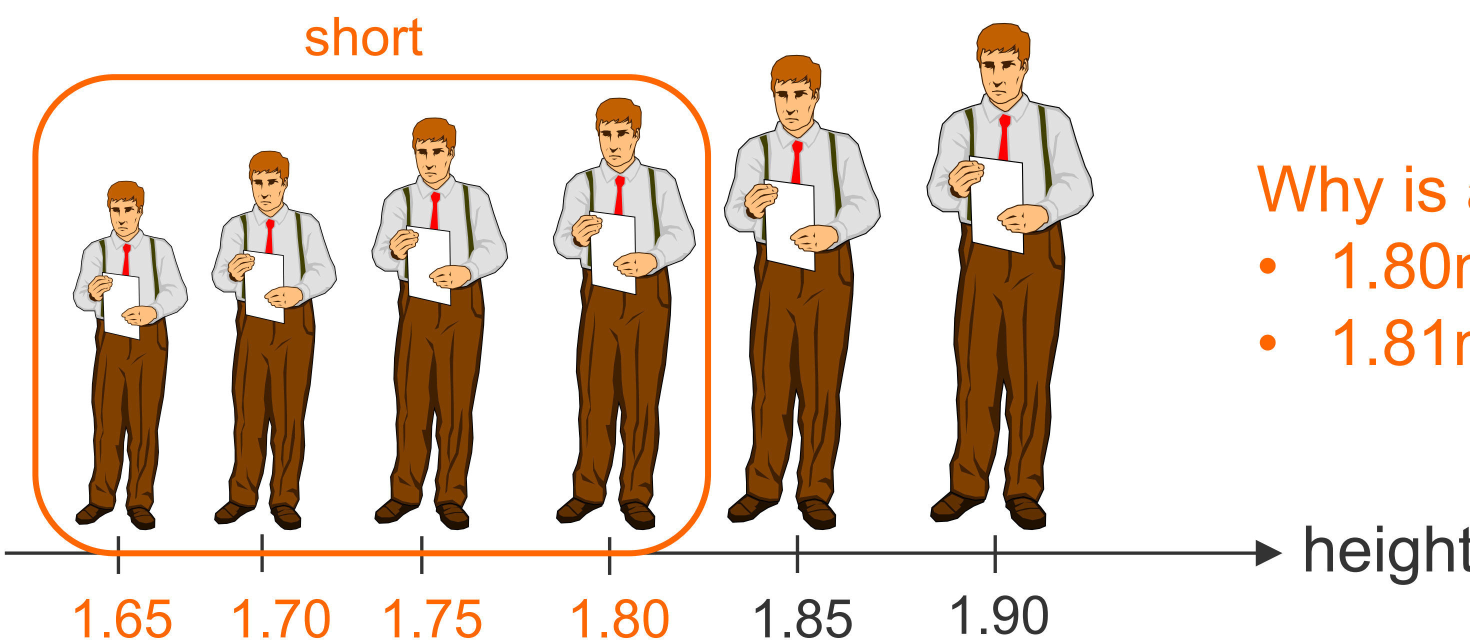



| height | short? |
|---|---|
| 1.65 | 1 |
| 1.70 | 1 |
| 1.75 | 1 |
| 1.80 | 0 |
| 1.85 | 0 |
| 1.90 | 0 |

# Classical vs. Fuzzy Sets

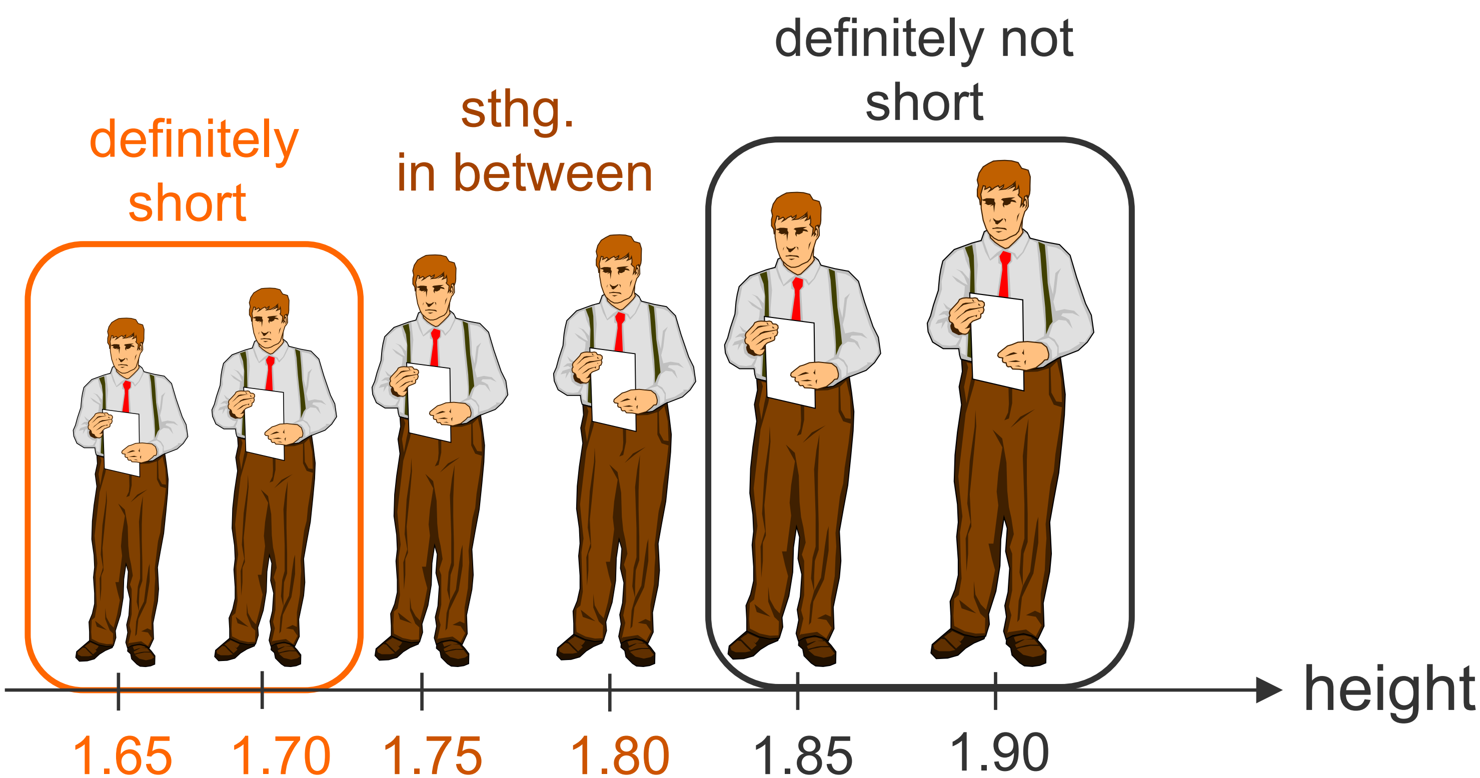
- When is a man short?
- Classical Set Theory: Either short or not short.
  E.g.: set of short men S= {m | height(m) ≤ 1.80}




Why is a man of
- 1.80m short
- 1.81m not short?

# Classical vs. Fuzzy Sets

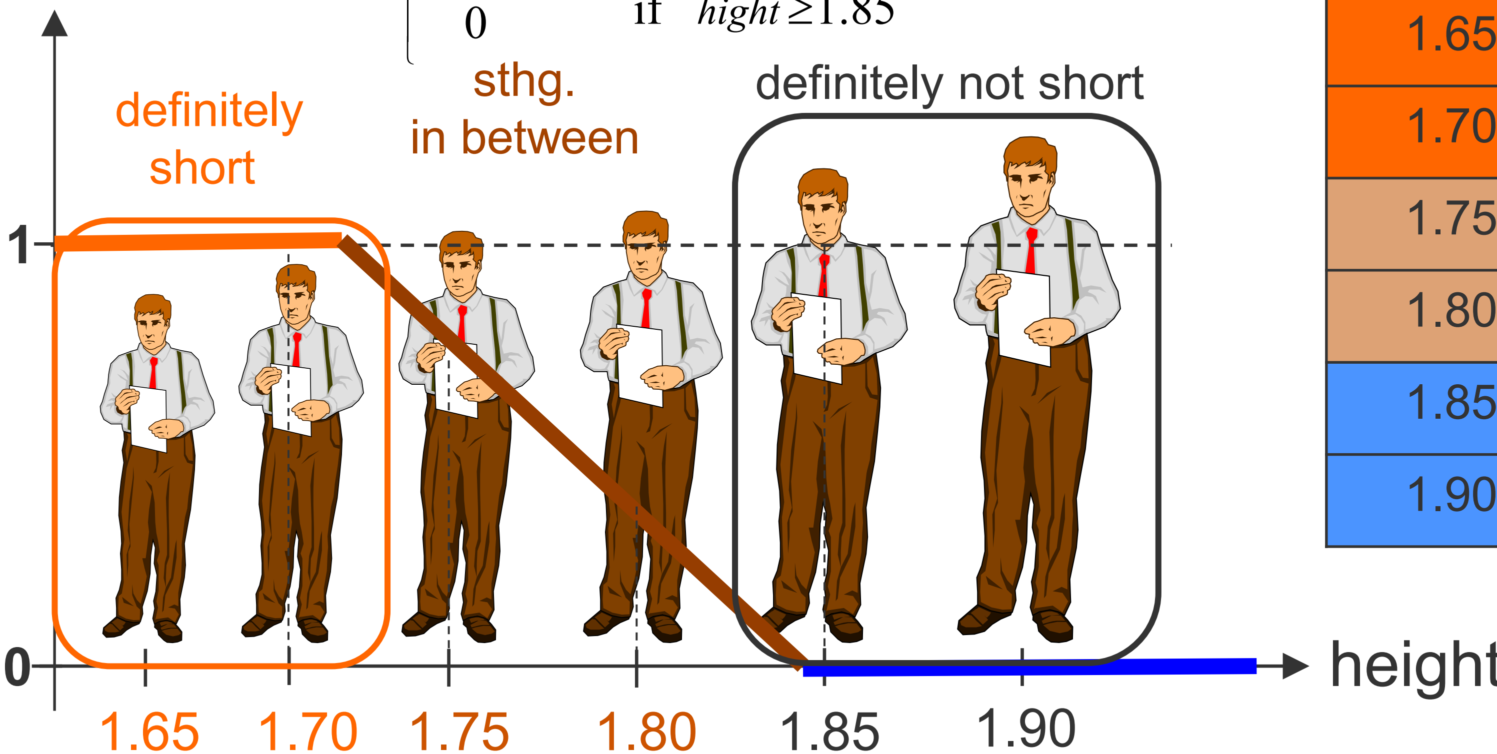
- Fuzzy sets have unsharp boundaries:



# Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(hight) = \begin{cases} 1 & \text{if } hight \le 1.70 \\ \frac{1.85 - hight}{1.85 - 1.70} & \text{if } 1.70 \le hight \le 1.85 \\ 0 & \text{if } hight \ge 1.85 \end{cases}$$

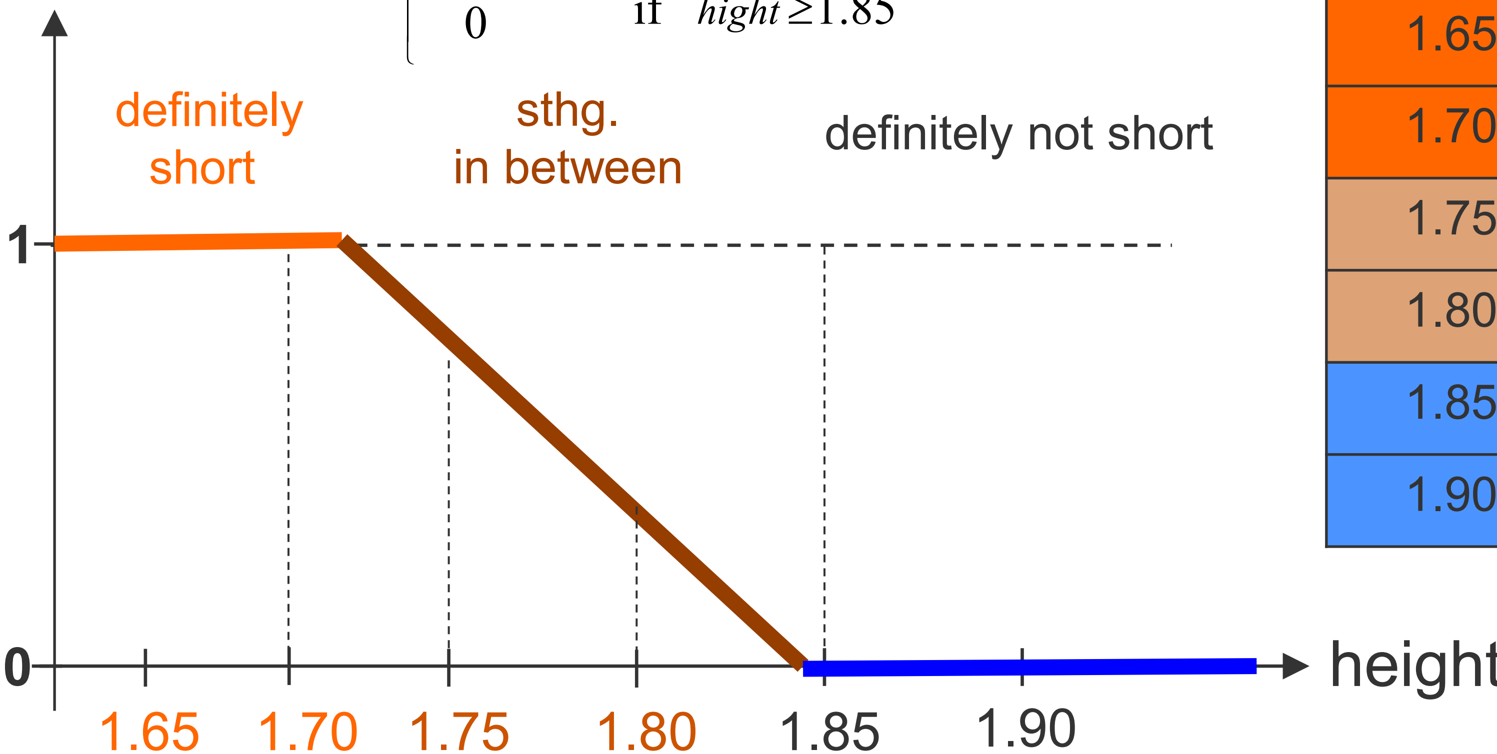



| *height* | *short?* |
|---|---|
| 1.65 | 1 |
| 1.70 | 1 |
| 1.75 | 2/3 |
| 1.80 | 1/3 |
| 1.85 | 0 |
| 1.90 | 0 |

# Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(hight) = \begin{cases} 1 & \text{if } hight \le 1.70 \\ \frac{1.85 - hight}{1.85 - 1.70} & \text{if } 1.70 \le hight \le 1.85 \\ 0 & \text{if } hight \ge 1.85 \end{cases}$$




| *height* | *short?* |
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| 1.65 | 1 |
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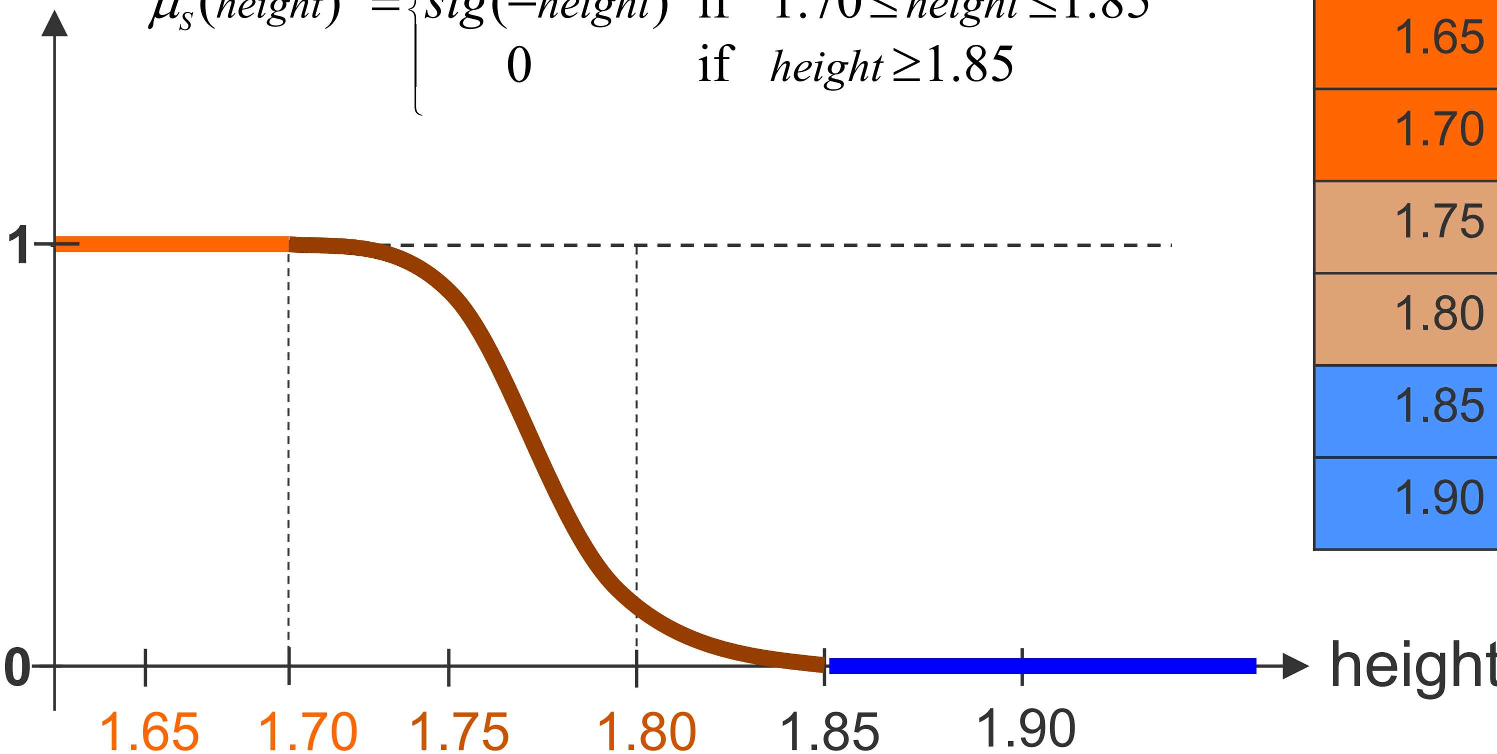
# Representation by membership function

Fuzzy sets, e.g.: fuzzy set of short men

$$\mu_S(height) = \begin{cases} 1 & \text{if } height \leq 1.70 \\ sig(-height) & \text{if } 1.70 \leq height \leq 1.85 \\ 0 & \text{if } height \geq 1.85 \end{cases}$$




| *height* | *short?* |
|---|---|
| 1.65 | 1 |
| 1.70 | 1 |
| 1.75 | |
| 1.80 | |
| 1.85 | 0 |
| 1.90 | 0 |

# Classical vs. Fuzzy Sets

- A classical set can be seen as a special case of a fuzzy set, where the fuzziness of the set boudary is infinitely small.
- Classical sets are also called crisp sets.

# Exercise: Fuzzy Sets for Size of People

- Draw fuzzy sets for short, medium and tall men; use trapezoidal membership functions.
- Here are the restrictions:
  - Men below 1.60 are definitely short
  - Men taller than 175 are definitely not short
  - Men taller than 190 are definitely tall
  - Men smaller than 180 are not tall
  - Men between 170 and 185 are medium
  - Men below 165 are not medium
  - Men taller than 190 are not medium

# FUZZY SET THEORY

# Fuzzy Set Theory

## Operations on Fuzzy Sets:

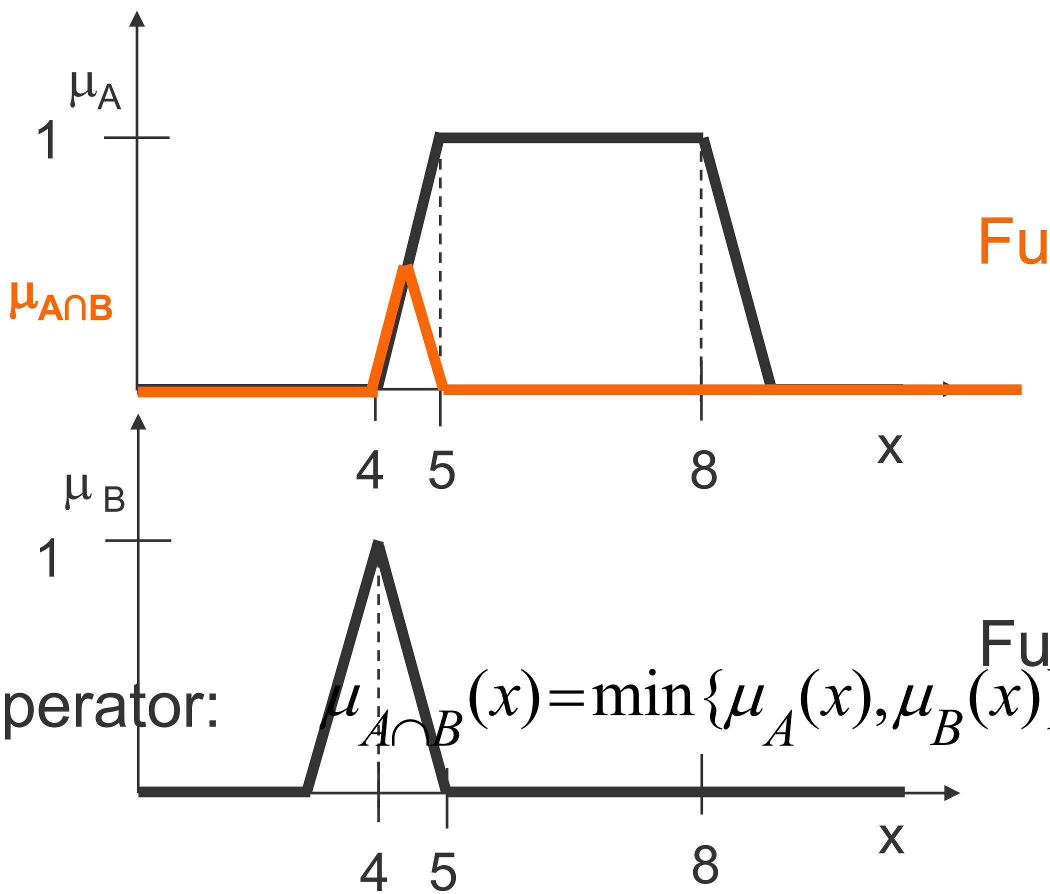
For Fuzzy Sets we can define operations

- ***intersection***,
- ***union***
- ***negation***

… analogue to classical sets.

# Operations with Fuzzy Sets

## Intersection:



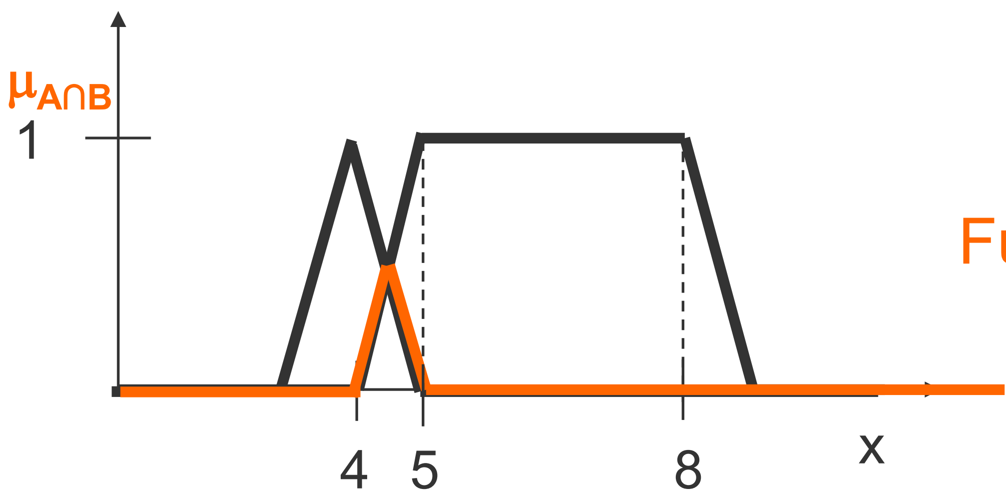

Fuzzy Set A∩B

Fuzzy Set B

Minimum Operator: $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

# Operations with Fuzzy Sets

## Intersection:

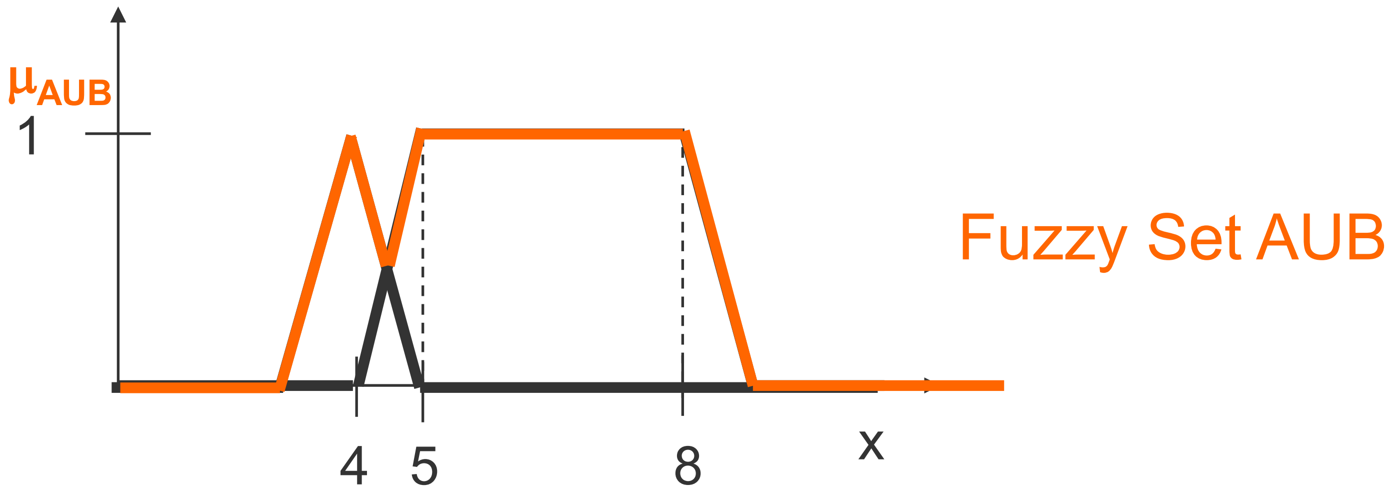



Fuzzy Set A∩B

Minimum Operator: $$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

# Operations with Fuzzy Sets

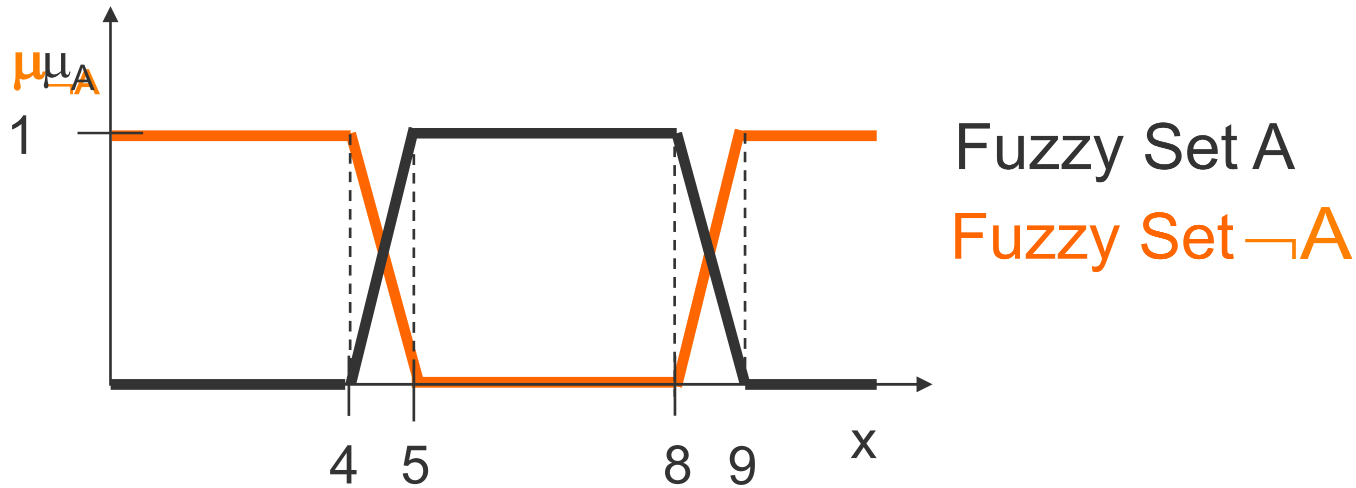
## Union:




Maximum Operator: $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$

# Operations with Fuzzy Sets

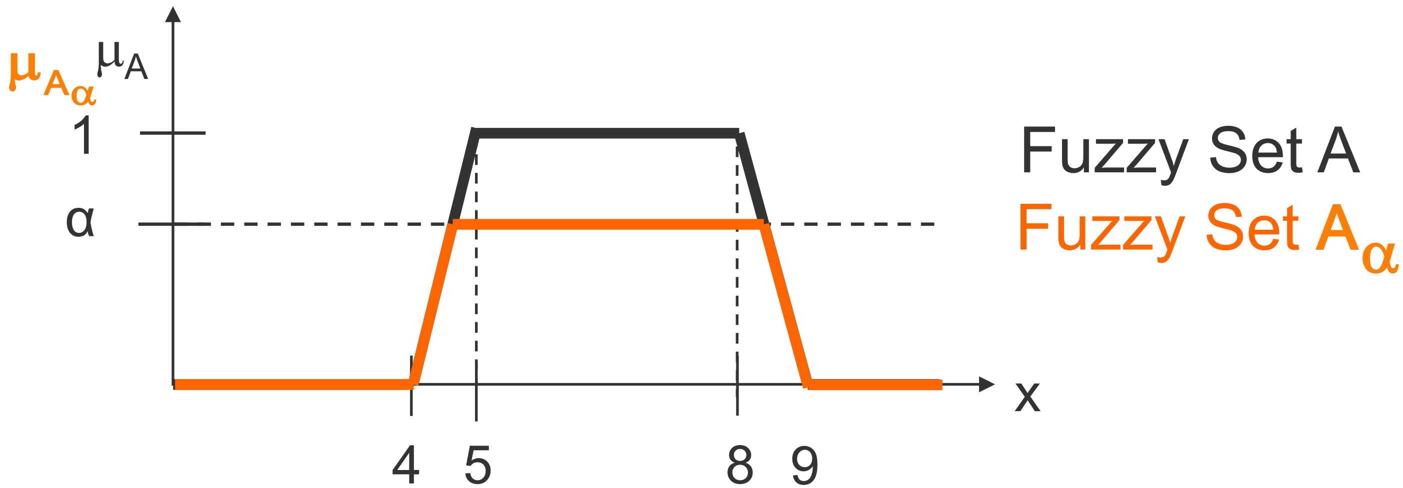
## Negation:




Complement Operator: $\mu_{\neg A}(x) = 1 - \mu_A(x)$

# Operations with Fuzzy Sets

## Alpha-cut:




α-Cut Operator: $\mu_{A_\alpha}(x) = \min\{\mu_A(x), \alpha\}$

# Exercise: Fuzzy Sets for Size of People (2)

- Draw the following fuzzy sets of people:
  - NOT short
  - NOT medium
  - NOT tall
  - Short UNION(OR) NOT tall
  - NOT short INTERSECTION(AND) NOT tall

- Is (NOT Short INTERSETION(AND) NOT tall) = medium?

# FUZZY LOGIC

# Fuzzy Logical Operators

- They modify or combine fuzzy logical statements.
  - E.g.: AND, OR, NOT, …
- They are operations on membership degrees:
  - AND: minimum, $\mu_{A \wedge B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$
  - OR: maximum, $\mu_{A \vee B}(x,y) = \max\{\mu_A(x), \mu_B(y)\}$
  - NOT: complement $\mu_{\neg A}(x) = 1 - \mu_A(x)$
- Note: There are serveral possibilities to define fuzzy logic operators! We use the above.

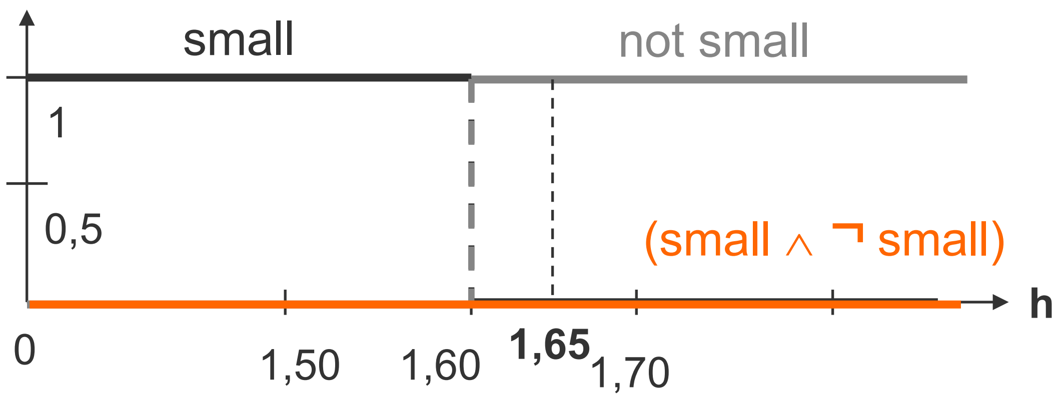
# Fuzzy Logic „Paradox"

In classical logic, a statement and its negation cannot be true at the same time:

$$(s \ni \neg s) = 0$$

„Tertium non datur"
(law of the excluded middle)

Example: Classical statement s=„Bob is small",
where *small* is specified by the following crisp set:



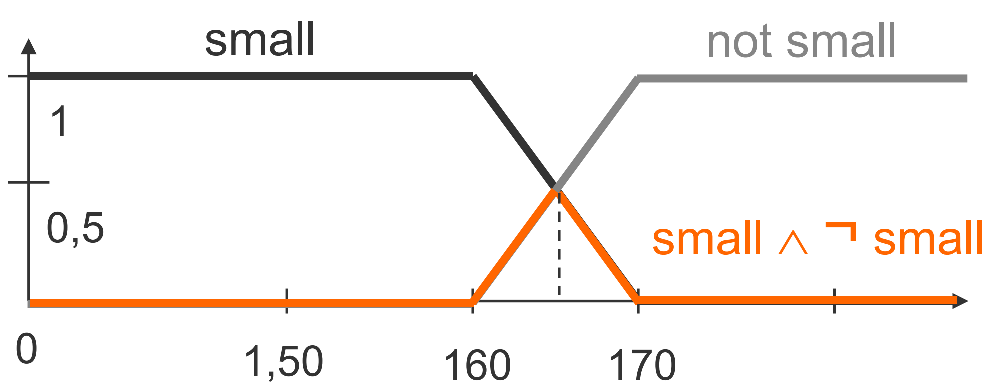

If height(Bob)=1.65, then $(s \wedge \neg s) = \min\{0,1\}=0$.

# Fuzzy Logic „Paradox“

In fuzzy logic, a statement and ist negation can both be (partially) true at the same time:

(s ∍ ¬ s) ≠ 0 for some s

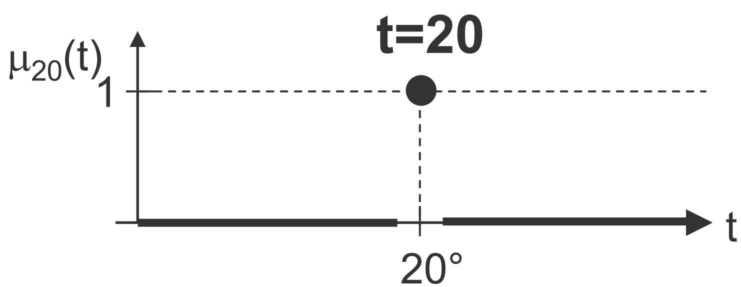
Example: Fuzzy statement s= „Bob is small“,
where small is specified by the following fuzzy set:




If height(Bob)=1.65, then (s ∧ ¬ s) = min{0.5,0.5}=0.5

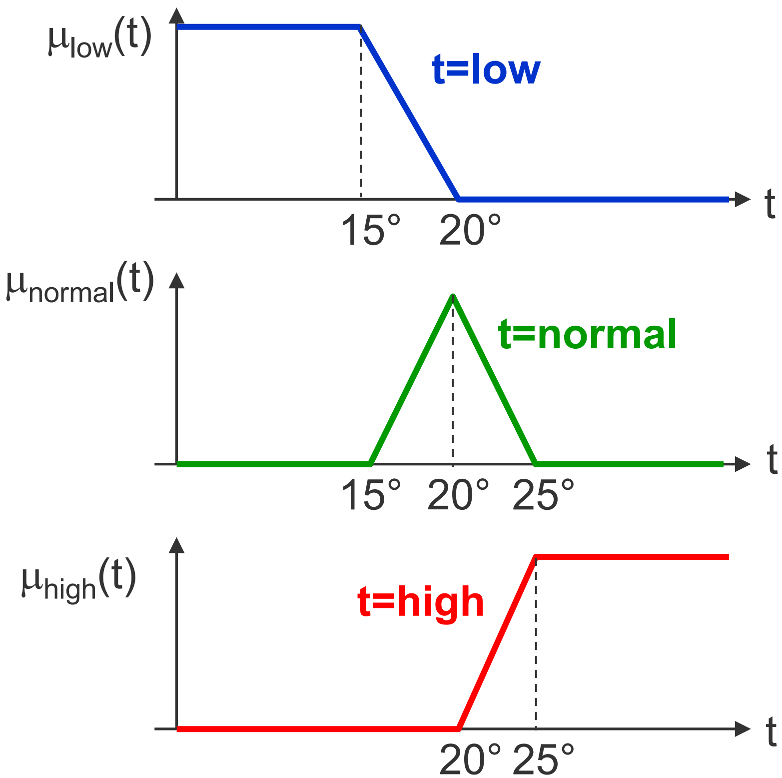
# Classical vs. Linguistic Variables

Example: Classical variable «**temperature**» (t).
t takes exact values in the interval **[-50,50]**,e.g., t=20:

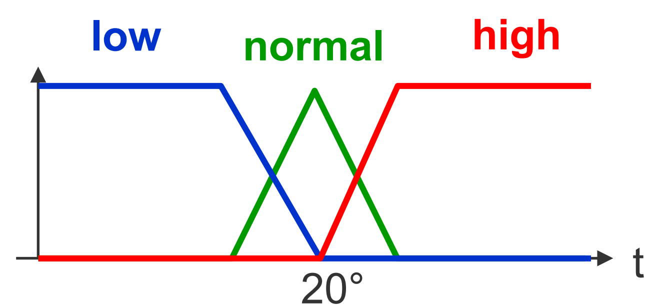


# Classical vs. Linguistic Variables

Example: Linguistic variable «**temperature**» (t).
t takes the fuzzy values ***low, normal, high***,*e.g., t=low.*
Fuzzy values are defined as **Fuzzy Sets**:



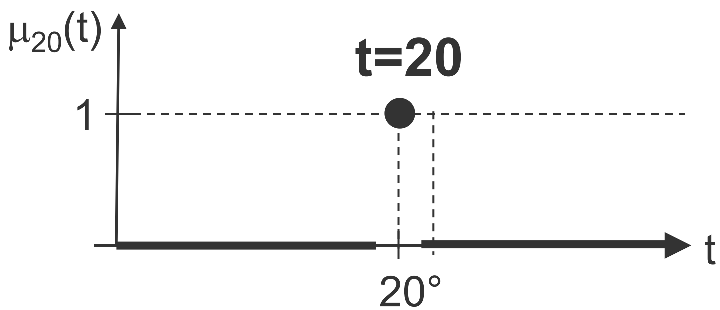

In one graphic:



# Classical Logical Statements

The possible truth values of an exact statement are:
1 (True) or 0 (False).

**Example:** Exact statement s=«The temperature is 20°C.»




«Temperature» is a classical variable (t). Takes the value t=20.

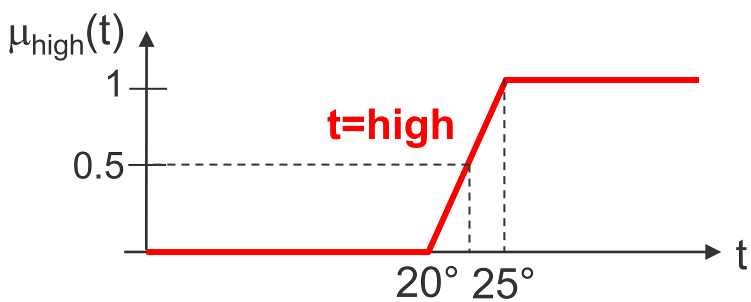
Assume the temperature is 22.5°C.

Then the truth value of s is 0.

# Fuzzy Logical Statements

The possible truth values of a fuzzy statement are 1 (True), 0 (False), and every value in between.

**Example**: Fuzzy statement s=«The temperature is high.»



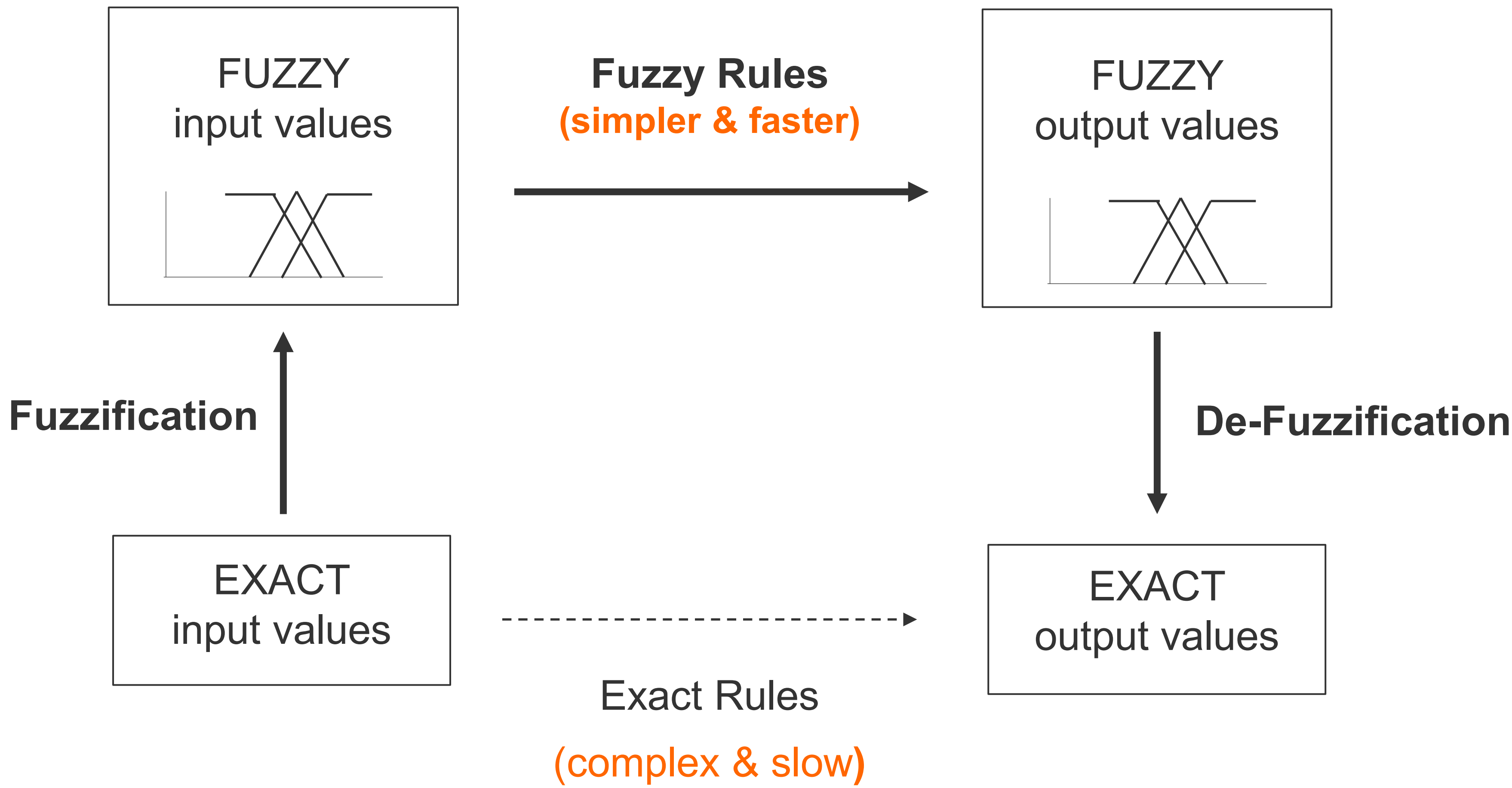

«Temperature» is a linguistic valriable (t). Takes the value t=high.

Assume the temperature is 22.5°C.
Then the truth value of s is 0.5.

The truth value of a fuzzy statement is also called truth degree. The truth degree indicates the degree of compatibility of the exact value 22.5°C with the fuzzy statements s.

# APPLICATIONS OF FUZZY LOGIC

# Designing a Fuzzy Controller (Procedure)




Input variables

Output variables

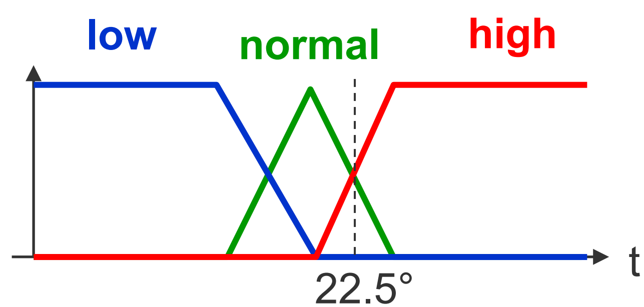
# Fuzzification

- Transformation of exact variables to linguistic variables, and
- Transformation of exact values to fuzzy values (fuzzy sets).

Example: Fuzzification of variable «temperature»:

$$t \in [-50,50] \rightarrow t \in \{low,\ normal,\ high\}$$

$$t = 22.5°C \rightarrow \{\mu_{low}(t)=0,\ \mu_{normal}(t)=0.5,\ \mu_{high}(t)=0.5\}$$



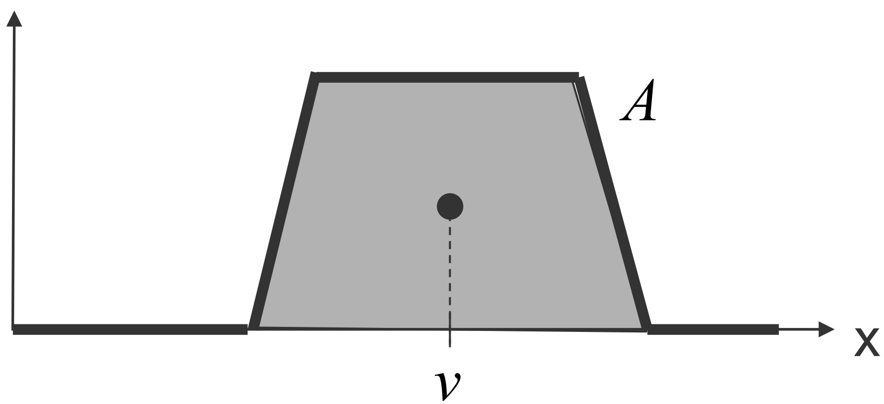
# Defuzzification

= Transformation of a fuzzy set to an exact value (number).

Different possible methods, e.g.,

- Center of gravity method
- Maximum method
- Weighted average method

Example: Centre of gravity method (Sugeno 1985, most commonly used):




$$v = \frac{\int \mu_A(x) \cdot x \, dx}{\int \mu_A(x) \, dx}$$

Disadvantage: Computationally difficult for complex membership functions.

# Example: Fuzzy Logic Controller

- **Problem:** Car heating system
  - The heating systems of a car should keep the temperature constant.
  - The heating power that is necessary depends on the **temperature** and the **air humidity** in the car:
    - The *higher* the temperature, the *lower* must be the heating power.
    - The *lower* the temperature, the *higher* must be the heating power.
    - The humidity interacts with temperature.
  - Sensors show the current temperature and humidity.

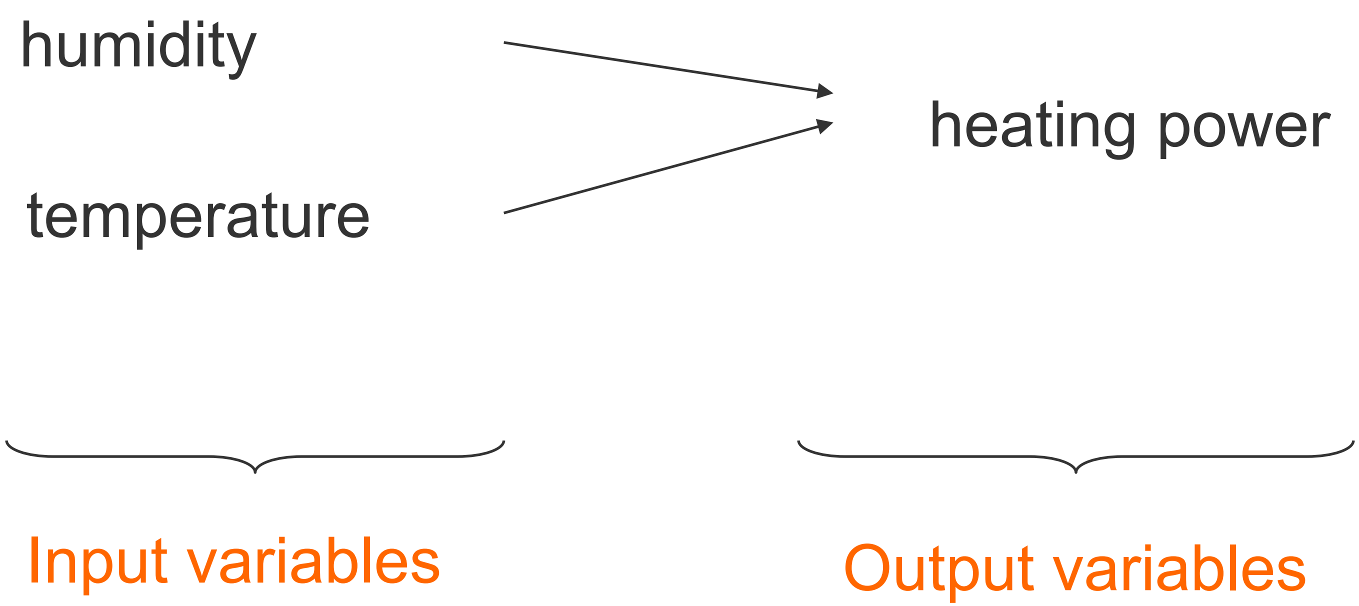
# Fuzzy Logic Controller „Car heating system"

Steps to build the fuzzy controller

1. Specify Input and Output variables
2. Fuzzification of variables and values
3. Define fuzzy rules
4. Defuzzification

# Fuzzy Logic Controller „Car heating system”

**Step 1:** Specify Input and Output variables



# Fuzzy Logic Controller „Car heating system”

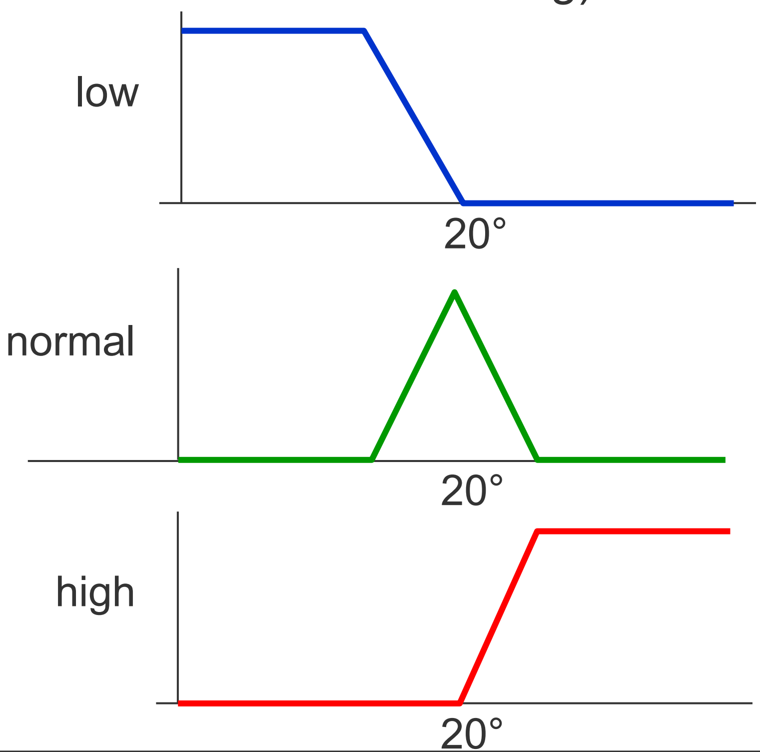
**Step 2a:** Fuzzification of variables and values:

- Determine linguistic variables:
  - Humidity: {low, medium, high}
  - Temperature: {low, normal, high}
  - Heating power: {low, normal, increased, high}
- Specify the fuzzy values of the linguistic variables as fuzzy sets
  - see next slide!

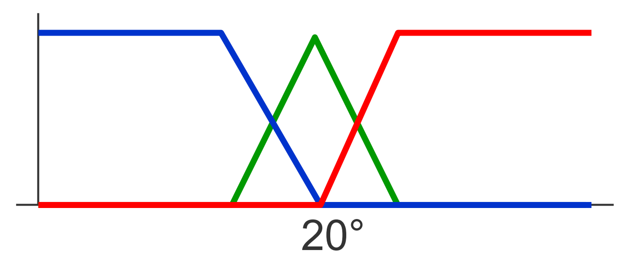
# Fuzzy Logic Controller „Car heating system"

## **Step 2b:** Fuzzification of variables and values:

- For each variable and each value a Fuzzy Set is defined.
- Here is the Fuzzification of temperature. It is assumed that the normal temperature is around 20° C (imagine it as the temperature that is adjusted at controller of the heating).

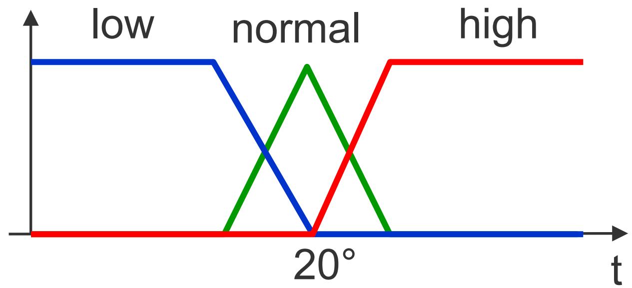
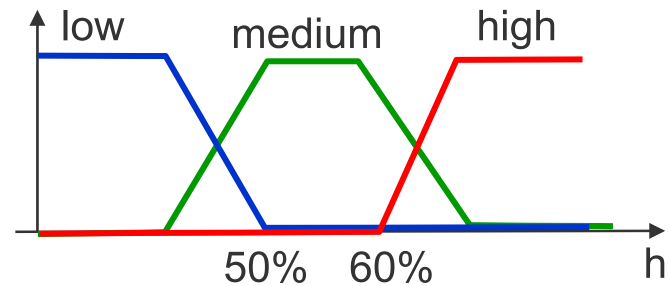
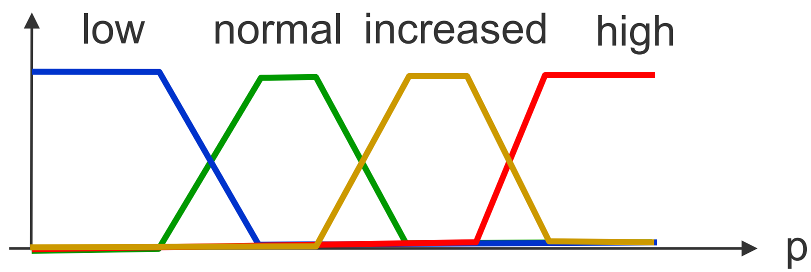



Fuzzy sets for temperature in one graphic



# Fuzzy Logic Controller „Car heating system”

## Step 2: Fuzzification of variables and values:

| | | |
|---|---|---|
| ***Input variables:*** | ***Temperature*** | ***Humidity*** |
| Fuzzy values: |   |   |
| ***Output variable:*** | ***Heating Power*** | |
| Fuzzy values: |   | |

# Fuzzy Logic Controller „Car heating system"

**Step 3:** Define fuzzy IF-THEN rules

- A fuzzy IF-THEN rule is NOT a logical implication, but can be thought of as a command.
- A set of Fuzzy IF-THEN rules maps linguistic variables to lingustic variabes (fuzzy function).
- Fuzzy IF-THEN rules describe the control of the system. They are similar to the experiences of an expert, who would formulate their knowledge in natural language terms.

# Fuzzy Logic Controller „Car heating system"

## **Step 3:** Define fuzzy IF-THEN rules

- Rule 1:

  IF Temperature = *low*
  THEN heating power is *increased*

- Rule 2:

  IF Temperature = *normal* AND humidity = *low*
  THEN heating power is *normal*

- Rule 3:

  IF Temperature = *normal* AND humidity = *high*
  THEN heating power is *high*

- Rule 4:

  IF Temperature = *high*
  THEN heating power is *low*

# Fuzzy Logic Controller „Car heating system"

**Step 3:** Define fuzzy IF-THEN rules … as decision table

Humidity / Temperature

| AND | **low** | **medium** | **high** |
| --- | --- | --- | --- |
| **low** | increased | increased | increased |
| **normal** | normal | | high |
| **high** | low | low | low |

White fields contain irrelevant cases

# Rule Application: „Car heating system”

Rule Application is performed in four steps:

1. Evaluate Antecedents:
   - For an exact input value, determine to which degree each antecedent is satisfied
   - Combine the degrees using the logical operators (AND in our example)
2. Evaluate Consequents:
   - The degree to wich an input variables $A_i$ is satisfied determines the degree to which the corresponding output variable $B_i$ holds (because IF-THEN rules are fuzzy functions). The result is the alpha cut of the output variable.
3. Aggregate Consequents:
   - Each rule gives one fuzzy set as a fuzzy output. Since all rules are valid, the fuzzy outputs may overlap (law of the excluded middle does not hold in general!). Combine them by OR to obtain a single fuzzy output value («aggregated output» ).
4. Defuzzify Aggregated Output

# Rule Application: „Car heating system"

## **Step 1:** Evaluate Antecedents

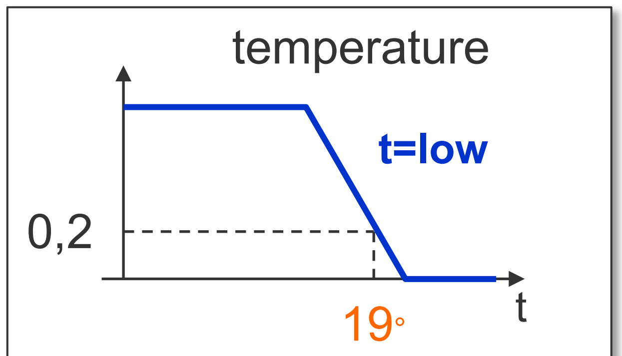
Assume the sensors have measured the following exact input values:

Temperature: t=19°
Humidity: h=45%

# Rule Application: „Car heating system”

## **Step 1:** Evaluate Antecedents

Rule 1: IF temperature = *low*
THEN heating power is *increased*



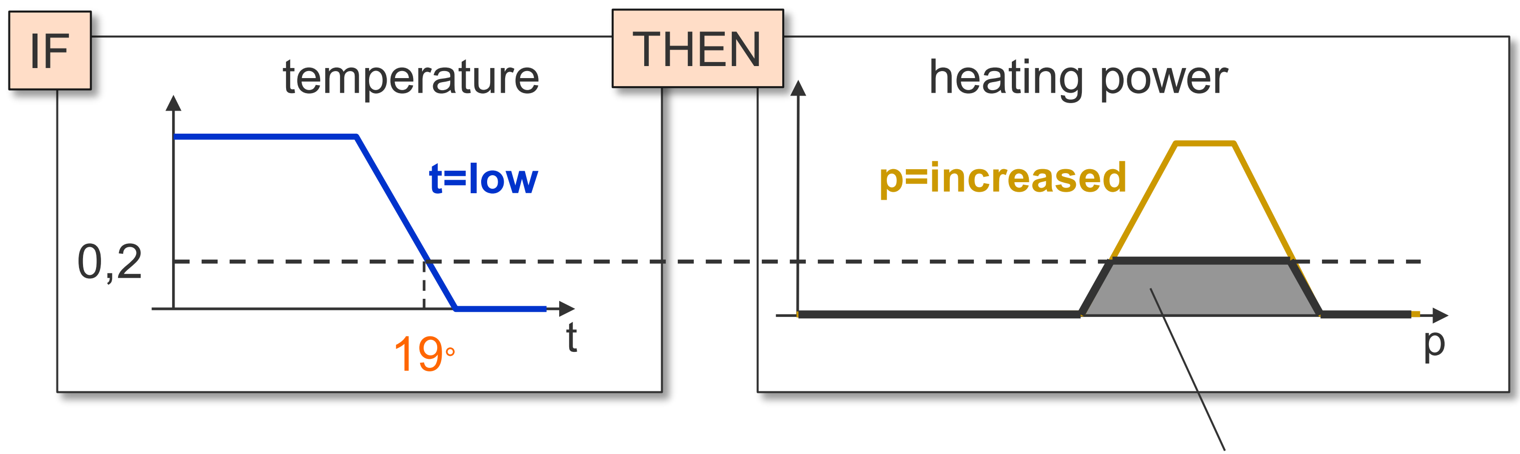

$$\mu_{t=low}(19°) = 0.2$$

# Rule Application: „Car heating system"

## **Step 2:** Evaluate Consequents

Rule 1:

IF temperature = *low*
THEN heating power is *increased*



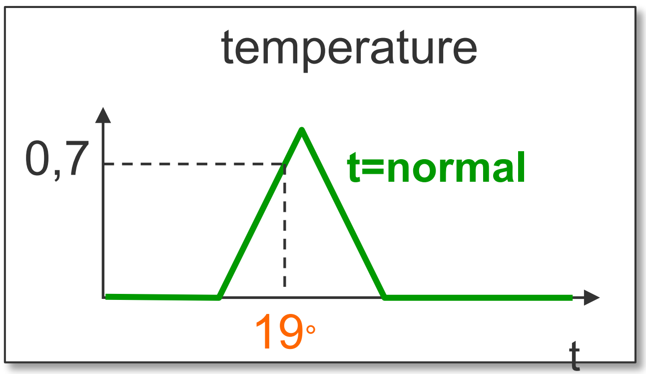

Output fuzzy set: $(\mu_{p=increased})_{0.2}$

( $\alpha$-cut of $\mu_{p=increased}$ with $\alpha = 0.2$ .)

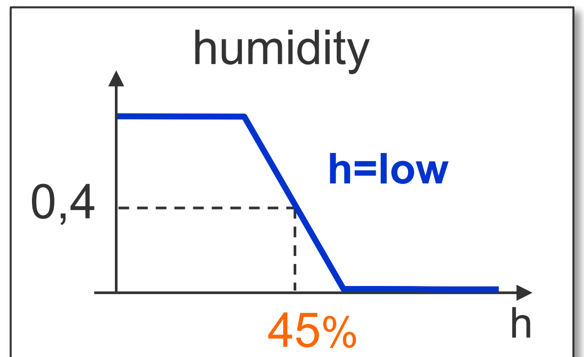
# Rule Application: „Car heating system”

## Step 1: Evaluate Antecedents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*




AND




$$\mu_{t=normal}(19°)=0.7$$

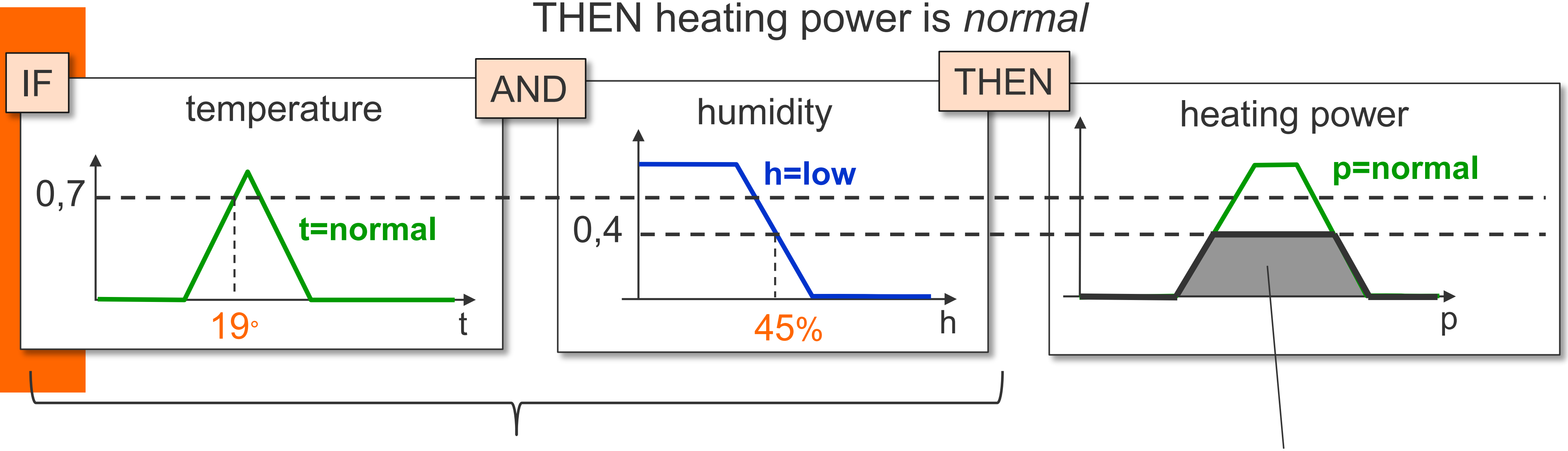
$$\mu_{h=low}(45\%)=0.4$$

Min-Operator for AND: $$\mu_{t=normal \wedge h=low}(19°,45\%)=\min\{0.7,\ 0.4\}=0.4$$

# Rule Application: „Car heating system"

## **Step 2:** Evaluate Consequents

Rule 2: IF temperature = *normal* AND humidity = *low*
THEN heating power is *normal*




Min-Operator for AND:

$$\mu_{t=normal \wedge h=low}(19°,45\%)=0.4$$

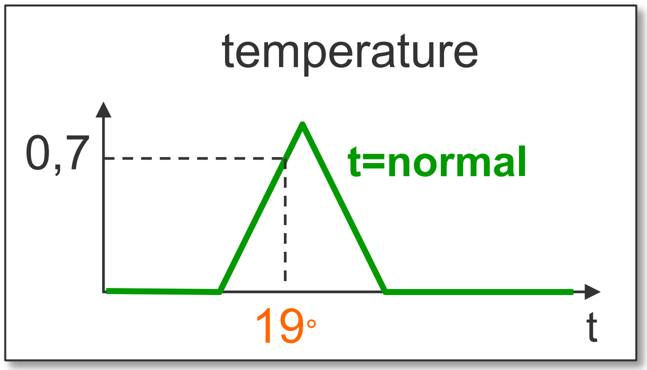
Output fuzzy set:

$$(\mu_{p=normal})_{0.4}$$

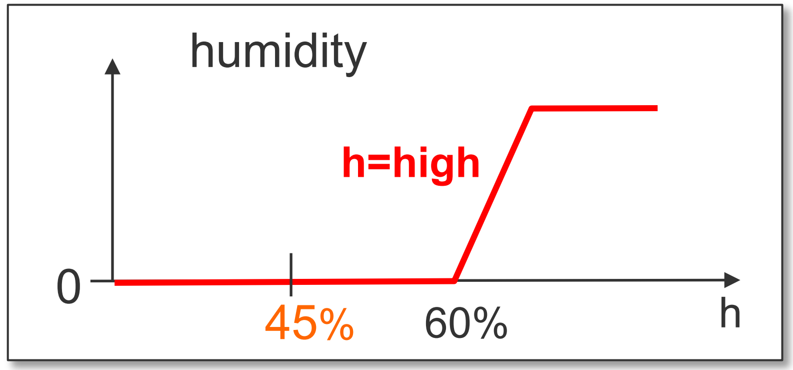
# Rule Application: „Car heating system"

## **Step 1:** Evaluate Antecedents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*




AND




$$\mu_{t=normal}(19°)=0.7$$

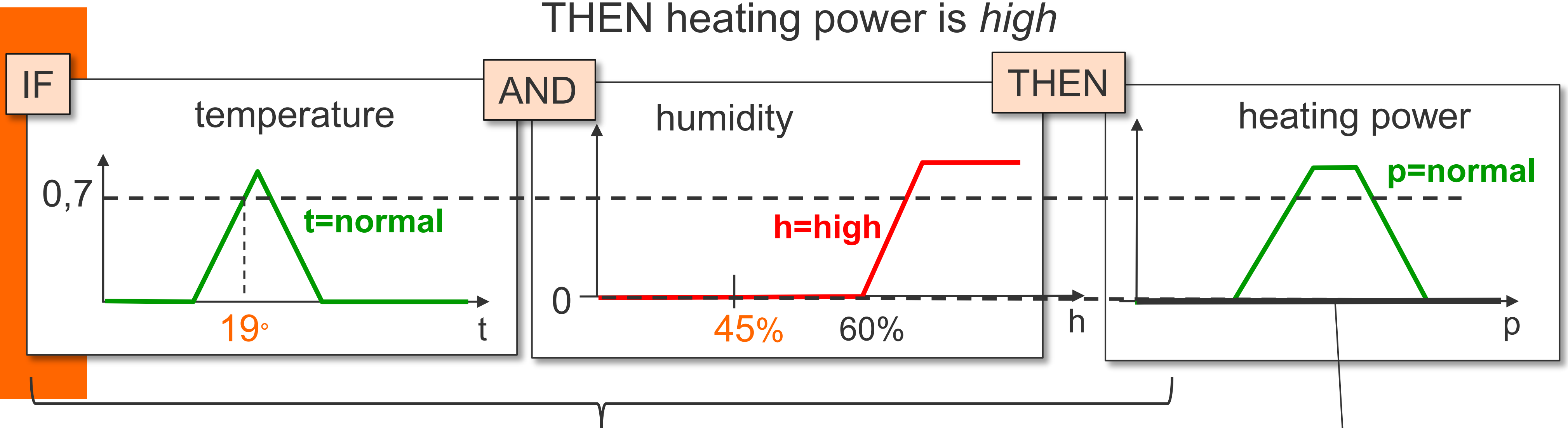
$$\mu_{h=high}(45\%)=0$$

Min-Operator for AND: $$\mu_{t=normal \wedge h=high}(19°,45\%)=\min\{0.7,\ 0\}=0$$

# Rule Application: „Car heating system”

## **Step 2:** Evaluate Consequents

Rule 3: IF Temperature = *normal* AND humidity = *high*
THEN heating power is *high*




Min-Operator for AND:

$$\mu_{t=normal \wedge h=high}(19°,45\%) = 0$$

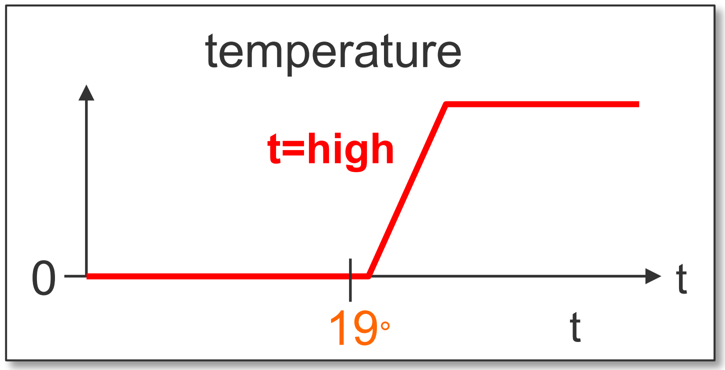
Output fuzzy set:

$$(\mu_{p=normal})_0 \equiv 0$$

# Rule Application: „Car heating system”

## **Step 1:** Evaluate Antecedents

Rule 4: IF Temperature = *high*
THEN heating power is *low*

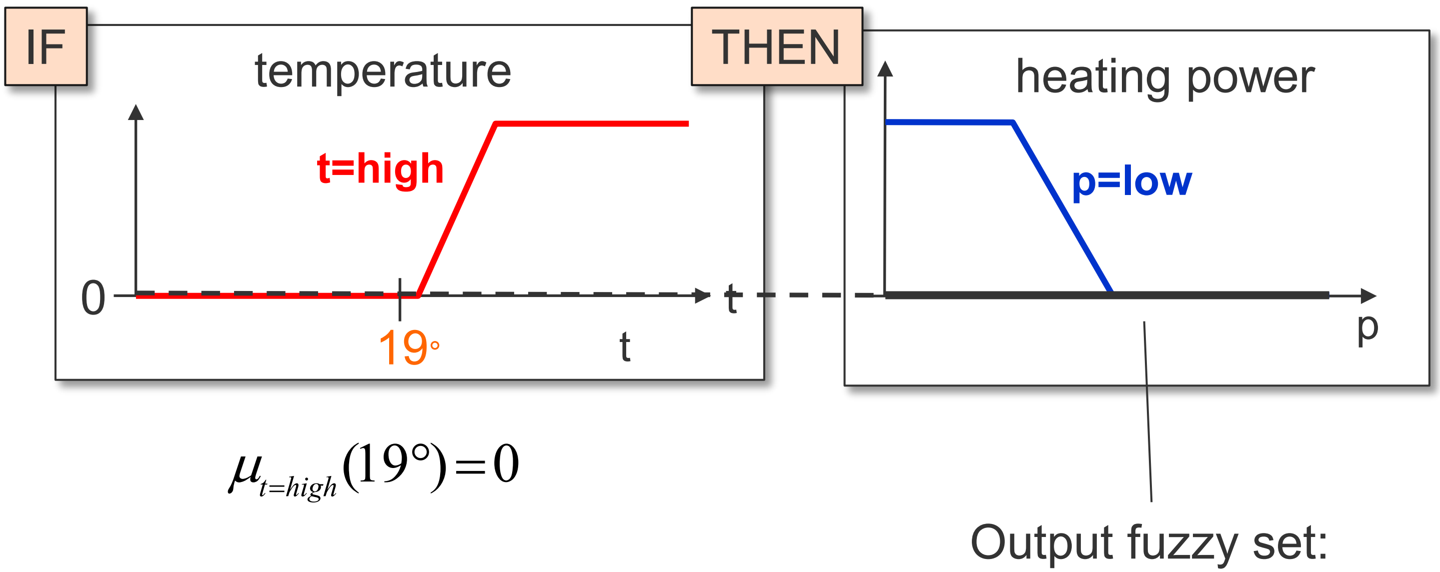



$$\mu_{t=high}(19°) = 0$$

# Rule Application: „Car heating system"

## **Step 2:** Evaluate Consequents

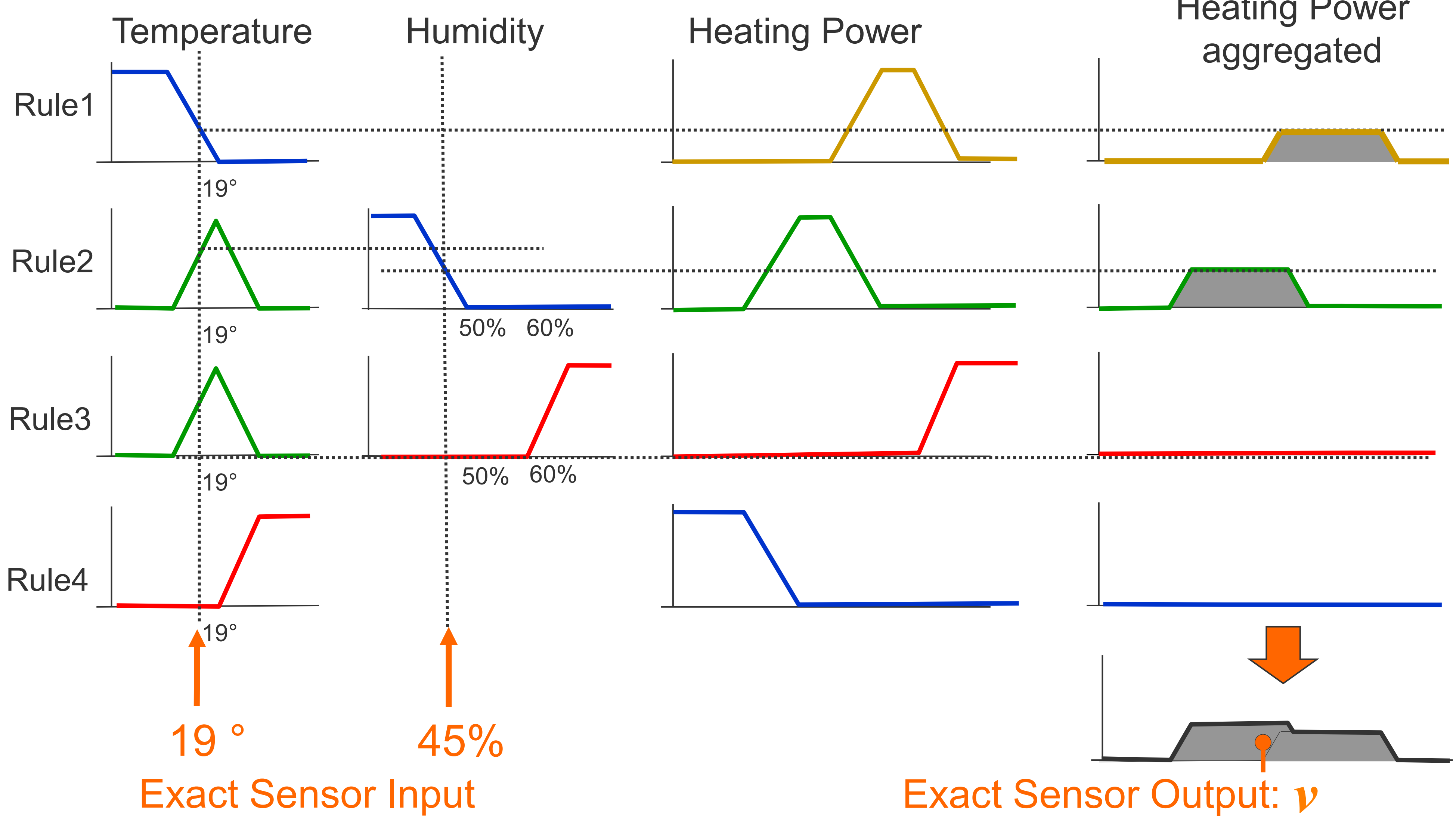
Rule 4: IF Temperature = *high*
THEN heating power is *low*




$$\mu_{t=high}(19°) = 0$$

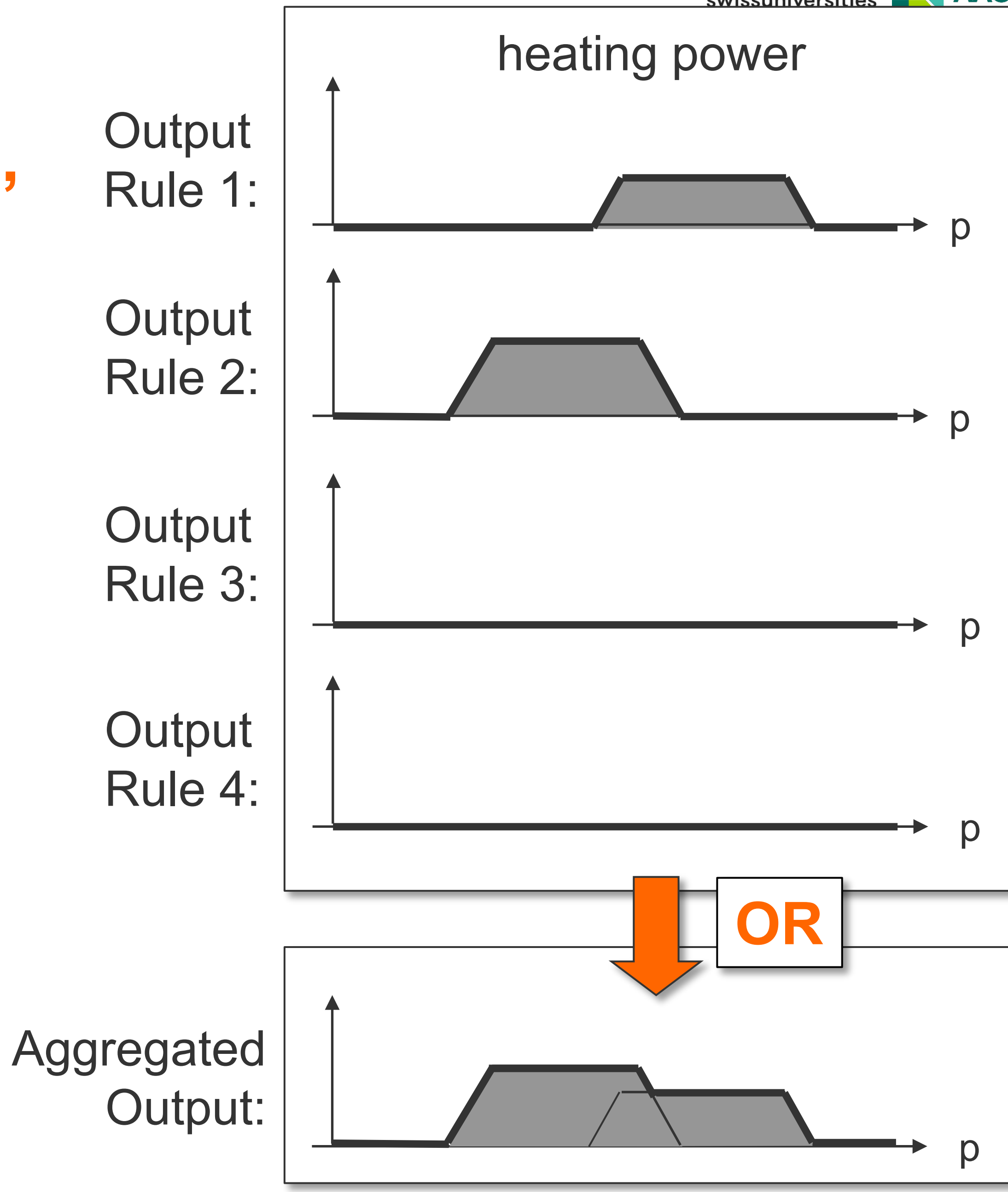
$$(\mu_{p=normal})_0 \equiv 0$$

# Rule Application: „Car heating system"



# Rule Application: „Car heating system”

**Step 3:** Aggregate Evaluated Consequents:



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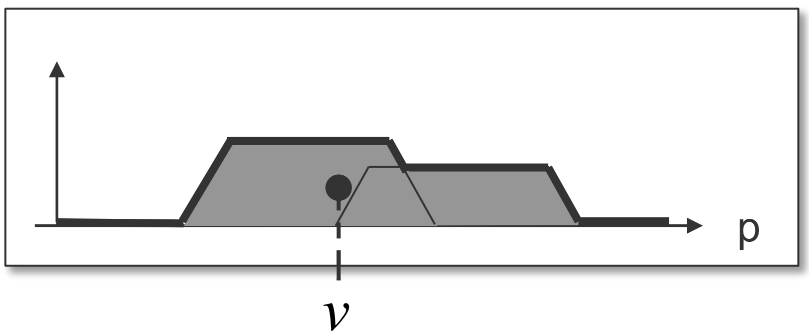
**Step 4:** Defuzzify aggregated output

Center of gravity method:




$$v = \frac{\int \mu_A(x) \cdot x \, dx}{\int \mu_A(x) \, dx}$$

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Main difference to exact reasoning:

Several rules can be active at the same time!
(Usually with different strengths.)