Advanced Topics in Software Engineering: Markov Chains

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Advanced Topics in Software Engineering
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- A stochastic process is a set of random variables $\{X(t), t \in T\}$.
- T is called the index set usually taken to represent time.
- Since we consider continuous time models $T = \mathbb{R}^{\geq 0}$, the set of non-negative real numbers.

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These paths are called sample paths or realisations of the stochastic process.

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$\{X(t)\}$ is a Markov process.
This implies that $\{X(t)\}$ has the Markov or memoryless property: given the value of $X(t)$ at some time $t \in T$, the future path $X(s)$ for $s > t$ does not depend on knowledge of the past history $X(u)$ for $u < t$, i.e. for $t_1 < \cdots < t_n < t_{n+1}$,

$$\Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \ldots, X(t_1) = x_1) = \Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n)$$

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$\{X(t)\}$ **is irreducible**.

This implies that all states in S can be reached from all other states, by following the transitions of the process. If we draw a directed graph of the state space with a node for each state and an arc for each event, or transition, then for any pair of nodes there is a path connecting them, i.e. the graph is strongly connected.

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$\{X(t)\}$ **is stationary**:
for any $t_1, \ldots t_n \in T$ and $t_1 + \tau, \ldots, t_n + \tau \in T$ $(n \geq 1)$, then the process's joint distributions are unaffected by the change in the time axis and so,

$$F_{X(t_1+\tau)\ldots X(t_n+\tau)} = F_{X(t_1)\ldots X(t_n)}$$

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$\{X(t)\}$ **is time homogeneous**:
the behaviour of the system does not depend on when it is observed. In particular, the transition rates between states are independent of the time at which the transitions occur. Thus, for all t and s, it follows that

$$\Pr(X(t+\tau) = x_k \mid X(t) = x_j) = \Pr(X(s+\tau) = x_k \mid X(s) = x_j).$$

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Note: by the Markov property, the sojourn times are memoryless.

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Thus, for $i \neq j, i, j \in S$, $\Pr(X(\tau + dt) = j \mid X(\tau) = i) = q_{ij} dt + o(dt)$ where the $q_{ij} = q_i p_{ij}$, by the decomposition property.

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The transition probability p_{ij} is the probability, given that a transition out of state i occurs, that it is the transition to state j. By the definition of conditional probability, this is $p_{ij} = q_{ij}/q_i$.

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By convention, the diagonal entries $q_{i,i}$ are the negative row sum for row i, i.e.

$$q_{i,i} = -\sum_{j=1, j\neq i}^{N} q_{i,j}$$

Steady state probability distribution

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From this probability distribution we will derive performance measures based on subsets of states where some condition holds.

Existence of a steady state probability distribution

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This distribution is reached when the initial state no longer has any influence.

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This is called the probability flux from state x_i to state x_j.

Global balance equations

In steady state, equilibrium is maintained so for any state the total probability flux out is equal to the total probability flux into the state.

$$\boxed{\underbrace{\boldsymbol{\pi_i} \times \sum_{x_j \in S, j \neq i} q_{ij}}_{\text{flux } \textbf{out} \text{ of } x_i} = \underbrace{\sum_{x_j \in S, j \neq i} (\boldsymbol{\pi_j} \times q_{ji})}_{\text{flux } \textbf{into } x_i}}$$

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(If this were not true the distribution over states would change.)

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Recall that the diagonal elements of the infinitesimal generator matrix $\mathbf{Q}$ are the negative sum of the other elements in the row, i.e.
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We can use this to rearrange the flux balance equation to be:

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Expressing the unknown values $\boldsymbol{\pi_i}$ as a row vector $\boldsymbol{\pi}$, we can write this as a matrix equation:

$$\boxed{\boldsymbol{\pi}\ \mathbf{Q} = 0}$$

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With these $n+1$ equations we can use standard linear algebra techniques to solve the equations and find the n unknowns, $\{\boldsymbol{\pi_i}\}$.

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- The CPUs execute in private memory for a random time before issuing a common memory access request. Assume that this random time is exponentially distributed with parameter λ.
- The common memory access duration is also assumed to be exponentially distributed, with parameter μ (the average duration of a common memory access is $1/\mu$).

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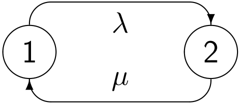
The system behaviour can be modelled by a 2-state Markov process whose state transition diagram and generator matrix are as shown below:



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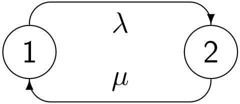

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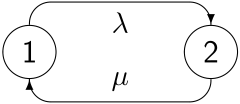

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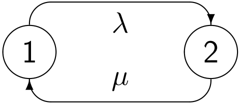

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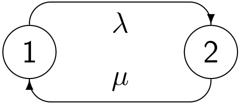

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From this we can deduce, for example, that the probability that the processor is executing in private memory is $\mu/(\mu+\lambda)$.

Solving the global balance equations

- In general the systems of equations will be too large to solve by hand.
- Instead we take advantage of linear algebra packages which can solve matrix equations of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- Here
  - $\mathbf{A}$ is an $N \times N$ matrix,
  - $\mathbf{x}$ is a column vector of N unknowns, and
  - $\mathbf{b}$ is a column vector of N values.

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   This problem is resolved by transposing the equation, i.e. $\mathbf{Q}^T\boldsymbol{\pi} = 0$, *where the right hand side is now a column vector of zeros, rather than a row vector.*

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We must also make the corresponding change to the "solution" vector $\mathbf{0}$, *to be a column vector with 1 in the last row, and zeros everywhere else. We denote this vector,* $\mathbf{e}_N$.

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Now we can use any linear algebra solution package, such as `MatLab` to solve the resulting equation:

$$\mathbf{Q}_N^T \boldsymbol{\pi} = \mathbf{e}_N$$

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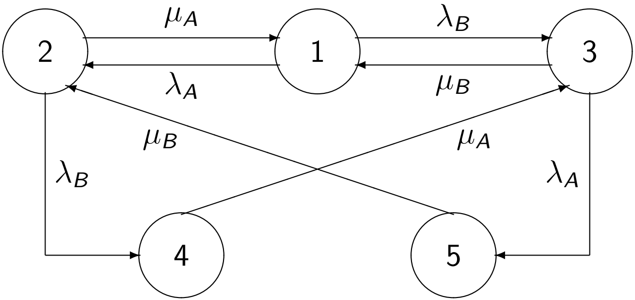
We assume that the processors have different timing characteristics, the private memory access of A being governed by an exponential distribution with parameter λ_A, the common memory access of B being governed by an exponential distribution with parameter μ_B, etc.

Example: state space

Now the state space becomes:

1. A and B both executing in their private memories;
2. B executing in private memory, and A accessing common memory;
3. A executing in private memory, and B accessing common memory;
4. A accessing common memory, B waiting for common memory;
5. B accessing common memory, A waiting for common memory;

Example: state space



Example: generator matrix

$$\mathbf{Q} = \begin{pmatrix} -(\lambda_A + \lambda_B) & \lambda_A & \lambda_B & 0 & 0 \\ \mu_A & -(\mu_A + \lambda_B) & 0 & \lambda_B & 0 \\ \mu_B & 0 & -(\mu_B + \lambda_A) & 0 & \lambda_A \\ 0 & 0 & \mu_A & -\mu_A & 0 \\ 0 & \mu_B & 0 & 0 & -\mu_B \end{pmatrix}$$

Example: modified generator matrix

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Example: steady state probability distribution

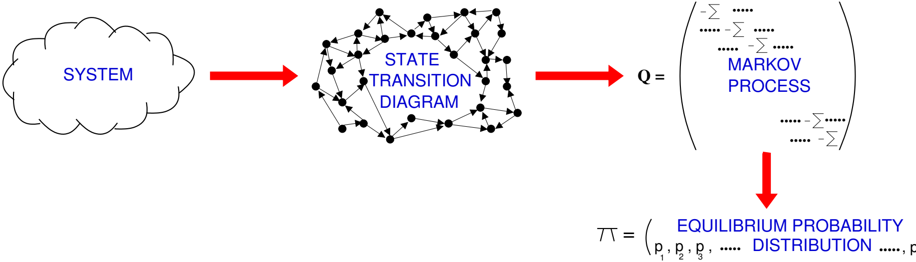
If we choose the following values for the parameters:

$$\lambda_A = 0.05 \qquad \lambda_B := 0.1 \qquad \mu_A = 0.02 \qquad \mu_B = 0.05$$

solving the matrix equation, and rounding figures to 4 significant figures, we obtain:

$$\boldsymbol{\pi} = (0.0693,\ 0.0990,\ 0.1683,\ 0.4951,\ 0.1683)$$

Deriving Performance Measures



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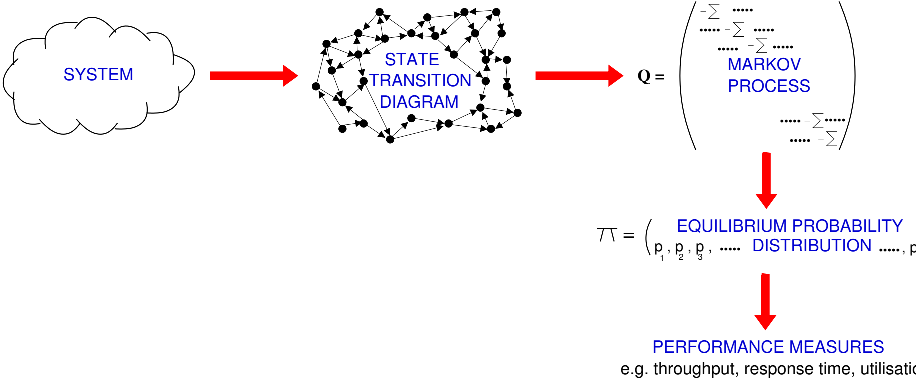

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- state-based measures, e.g. utilisation;
- rate-based measures, e.g. throughput;
- other measures which fall outside the above categories, e.g. response time.

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If we consider the multiprocessor example, the utilisation of the common memory, U_{mem}, is the total probability that the model is in one of the states in which the common memory is in use:

$$U_{mem} = \boldsymbol{\pi_2} + \boldsymbol{\pi_3} + \boldsymbol{\pi_4} + \boldsymbol{\pi_5} = 93.07\%$$

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For example, if we consider jobs waiting for the common memory to be queued in that subsystem, then the average number of jobs in the common memory, N_{mem}, is:

$$N_{mem} = (1 \times \pi_2) + (1 \times \pi_3) + (2 \times \pi_4) + (2 \times \pi_5) = 1.594$$

Rate-based measures

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This will be the product of the rate of the event, and the probability that the event is enabled, i.e. the probability of being in one of the states from which the event can occur.

Example: rate-based measures

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X_{mem} is thus calculated as:

$$X_{mem} = (\mu_A \times (\boldsymbol{\pi_2} + \boldsymbol{\pi_4})) + (\mu_B \times (\boldsymbol{\pi_3} + \boldsymbol{\pi_5})) = 0.0287$$

or, approximately one access every 35 milliseconds.

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For example, applying Little's Law to the common memory we see that

$$W_{mem} = N_{mem}/X_{mem} = 1.594/0.0287 = 55.54 \text{ milliseconds}$$

Assumptions

Stochastic Hypothesis

"The behaviour of a real system during a given period of time is characterised by the probability distributions of a stochastic process."

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- The Markov/memoryless assumption — future behaviour is only dependent on the current state, not on the past history — is a reasonable assumption for computer and communication systems, if we choose our states carefully.
- We generally assume that the Markov process is finite, time homogeneous and irreducible.

Exercise

- Consider the multiprocessor example, but with three processors, A, B and C sharing the common memory instead of two.
- List the states of the system, and draw the state transition diagram for this case.
- What is the difficulty in doing this and what further information do you need?

To be continued...

Advanced Topics in Software Engineering: Discrete Time Markov Chains

Prof. Michele Loreti

Advanced Topics in Software Engineering
Corso di Laurea in Informatica (L31)
Scuola di Scienze e Tecnologie

Discrete time Markov chains...

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Markov property is satisfied:

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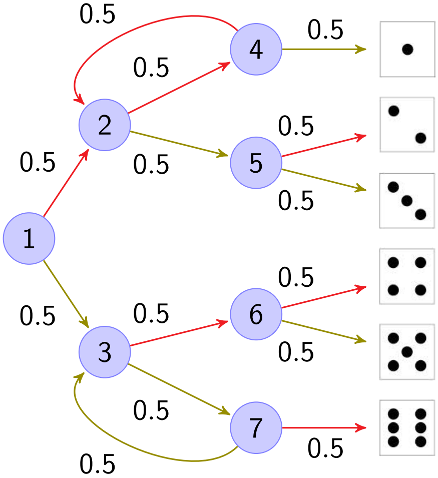
We can consider a state based view of a DTMC.

Example: Knut-Yao Algorithm

We can use a (fair) coin to mimic a dice (red edges stands for *head*, green for *tail*):



Questions. . .

Is the algorithm correct?

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On average, how many coin tosses are needed?

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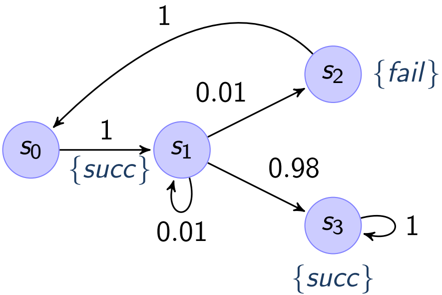
Transitions describe how system evolve from one state to the others in a discrete-time step.

Probabilities of transitions is given by discrete probability distributions.

Simple DTMC example

Modelling a very simple communication protocol:

- after one step a process starts trying to send a message;
- with probability 0.01, channel unready so wait a step;
- with probability 0.98, send message successfully and stop;
- with probability 0.01, send message fails, restart.



Discrete Time Markov Chains

A Discrete Time Markov Chain (DTMC) is a pair (S, P) where

- S is a set of states;
- $P : S \times S \rightarrow [0, 1]$ is a transition probability matrix:

$$\sum_{s' \in S} P(s, s') = 1 \qquad \text{(for all } s \in S\text{)}$$

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$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} s_0 \arrow[r, "1"] & s_1 \arrow[loop below, "0.01"] \arrow[r, "0.01"] \arrow[dr, "0.98"'] & s_2 \arrow[ll, bend right, "1"'] \\ & & s_3 \arrow[loop right, "1"] \end{tikzcd}$$

Definitions...

P is a stochastic matrix if and only if:

- $\forall s, s' \in S : P(s, s') \in [0, 1]$;
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A state $s \in S$ is absorbing if and only if:

$$P(s, s') = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{otherwise} \end{cases}$$

States and transitions

Current state of a DTMC can be rendered in terms of a probability distributions π over states ($Dist(S)$).

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Let $x \in [0,1]^S$ (that is a column vector associating each element in S with a value in $[0,1]$), a matrix-vector multiplication can be use to compute the reverse flow of x:

$$x' = Px$$

Example

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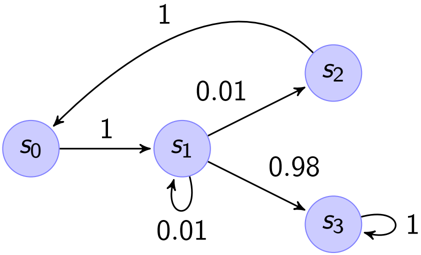

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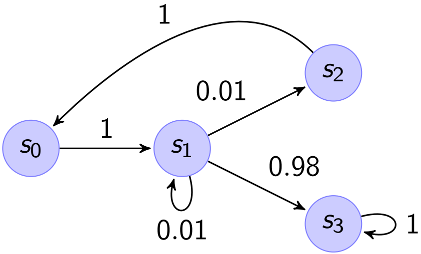

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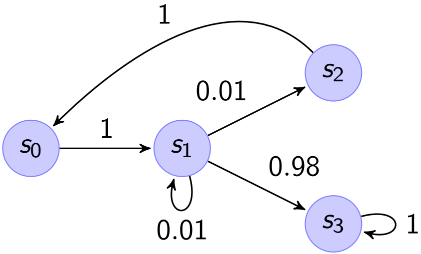

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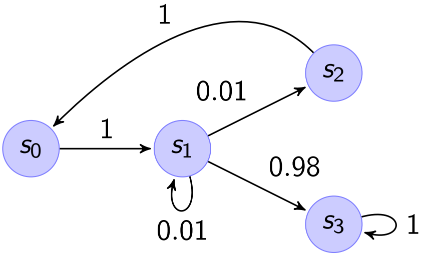

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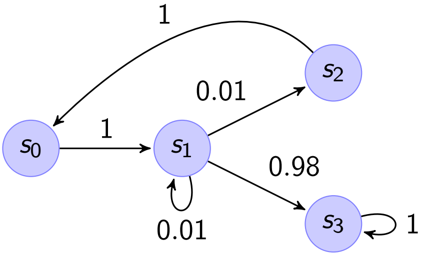

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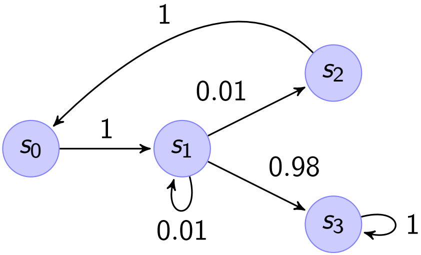

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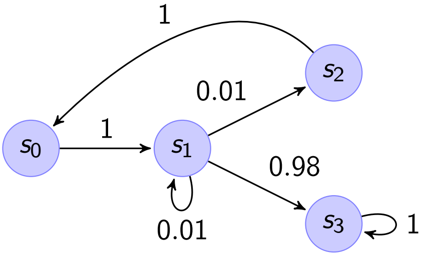

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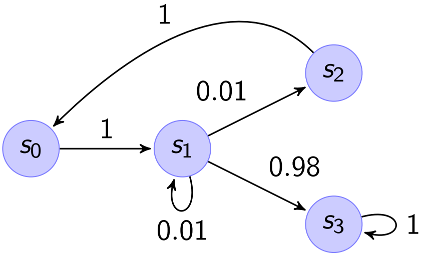

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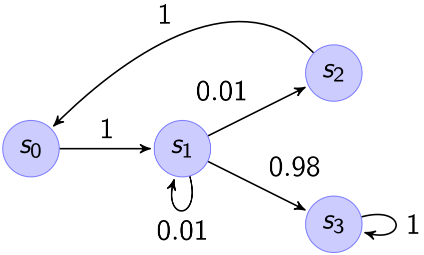

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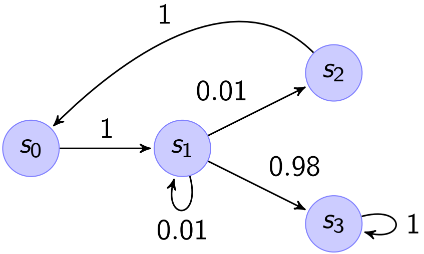

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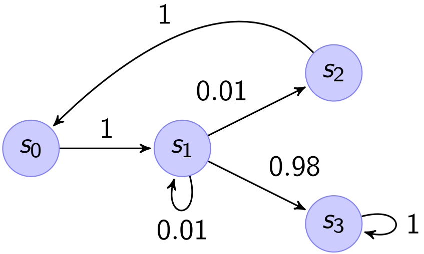

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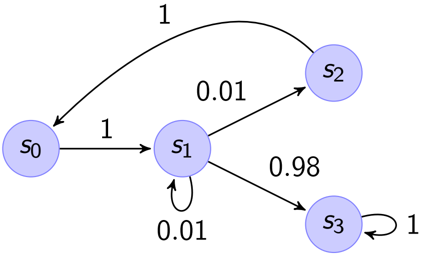

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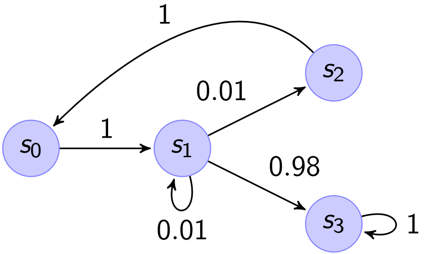

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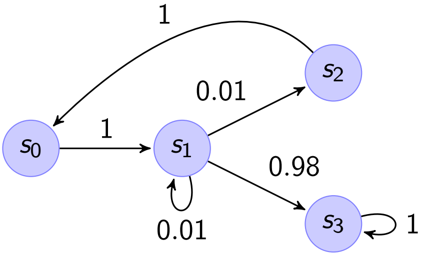

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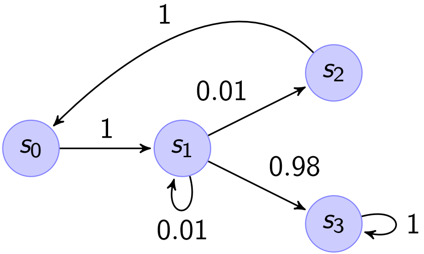

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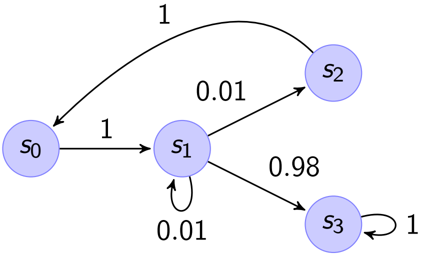

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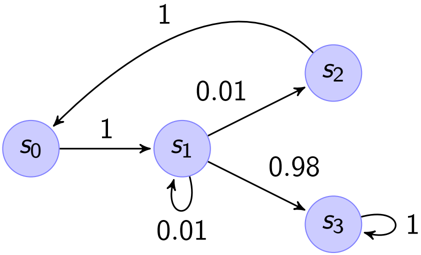

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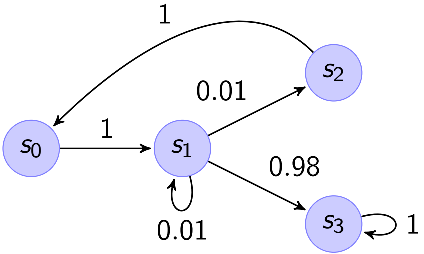

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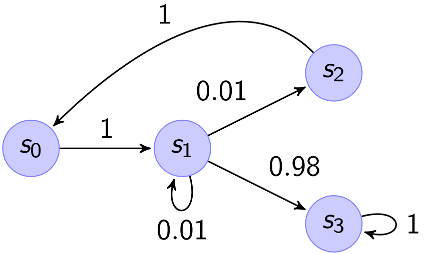

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Notation:

- $Path(s)$ denotes the set of paths starting in a state s;
- $Path_{fin}(s)$ denotes the set of finite path starting in a state s.

Paths and probabilities

To speak about the probability of paths, we need to define a probability space on $Path(s)$:

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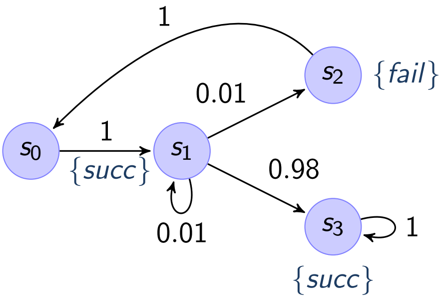
- $\Omega = Path(s)$
- $\Sigma_{Path(s)}$ is the σ-algebra on $Path(s)$ containing $Cyl(\omega)$, for any $\omega \in Paths_{fin}(s)\}$:

$$Cyl(\omega) = \{\omega' \in Path(s) | \omega \text{ is a prefix of } \omega'\}$$

- Probability measure Pr_s is defined as follows:
  - $Pr_s(Cyl(\omega)) = P_s(\omega)$ where

$$\begin{array}{rcl} P_s(s) & = & 1 \\ P_s(ss'\omega) & = & P(s,s') \cdot P_{s'}(s'\omega) \end{array}$$

Example...



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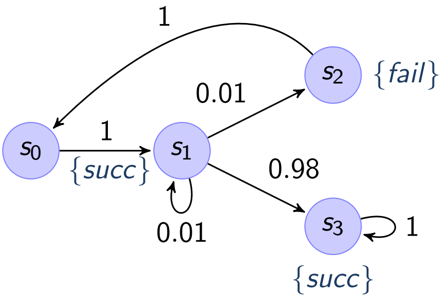
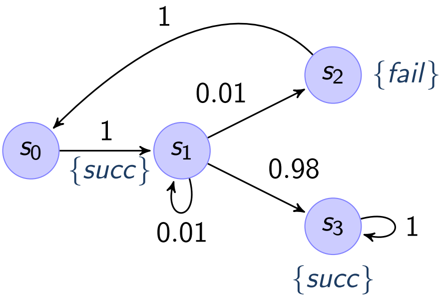


All the computations where sending immediately fails...

Example...



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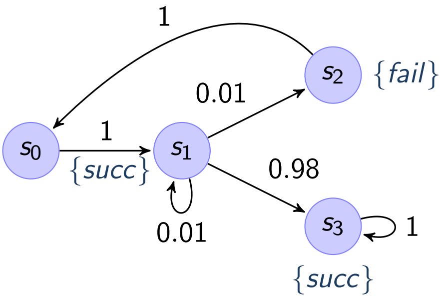

All the computations where sending immediately fails...

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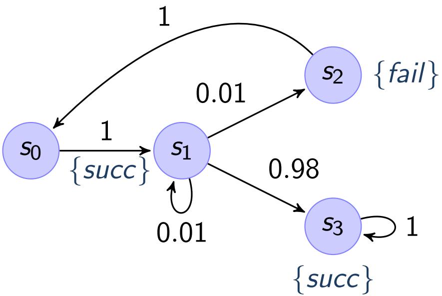
... this is $Cyl(s_0 s_1 s_2)$:

$$Pr_{s_0}(s_0 s_1 s_2) = P(s_0, s_1) \cdot P(s_1, s_2) = 1 \cdot 0.01 = 0.01$$

Example. . .



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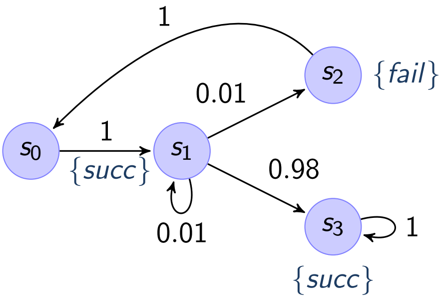

All the computations that are eventually successful with no failures

$$X = Cyl(s_0 s_1 s_3) \cup Cyl(s_0 s_1 s_1 s_3) \cup Cyl(s_0 s_1 s_1 s_1 s_3) \cup \cdots$$

$$Pr_{s_0}(X) = \sum_{i=0}^{\infty} 1 \cdot 0.01^i \cdot 0.98 = \frac{0.98}{0.99}$$

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- Example: an error never occurs.

Reachability probability

Let (S, P) be a DTMC and $T \subseteq S$, we let

$$ProbReach(s, T) = Pr(Reach(s, T))$$

where $Reach(s, T) = \{s_0 s_1 \ldots \in Path(s) | \exists i : s_i \in T\}$

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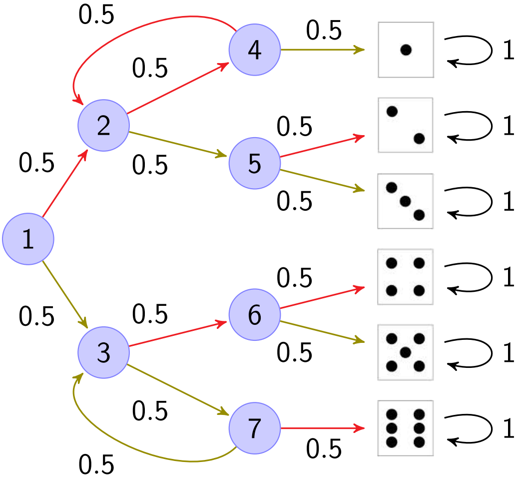
- $Pr(Reach(s, T)) = \sum_{\omega \in Path_{fin}(s,T)} Pr(Cyl(\omega))$.

Example: Knut-Yao Algorithm



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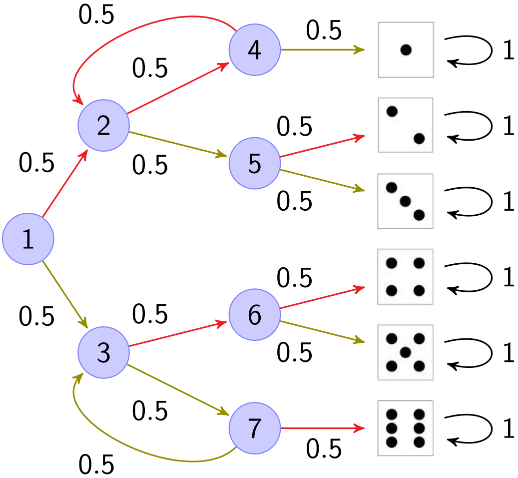
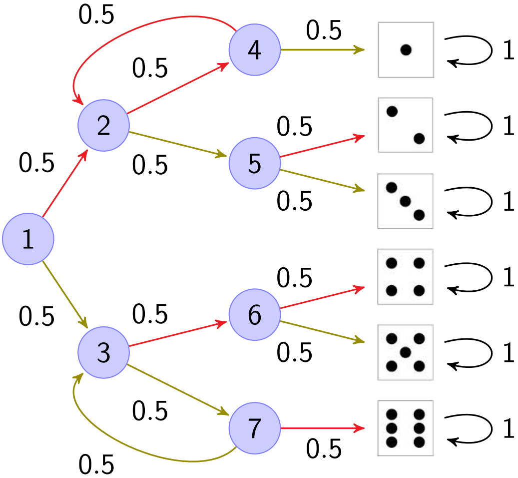


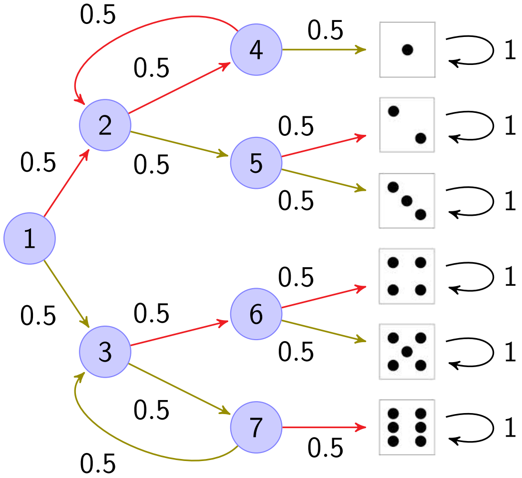
Example: Knut-Yao Algorithm

$ProbReach(s_1, ⚅) =$
$Pr(Cyl(s_1 s_3 s_7 ⚅))$



Example: Knut-Yao Algorithm

$$
\begin{aligned}
&ProbReach(s_1, \text{⚅}) = \\
&\quad Pr(Cyl(s_1 s_3 s_7 \text{⚅})) \\
&\quad + Pr(Cyl(s_1 s_3 s_7 s_3 s_7 \text{⚅}))
\end{aligned}
$$



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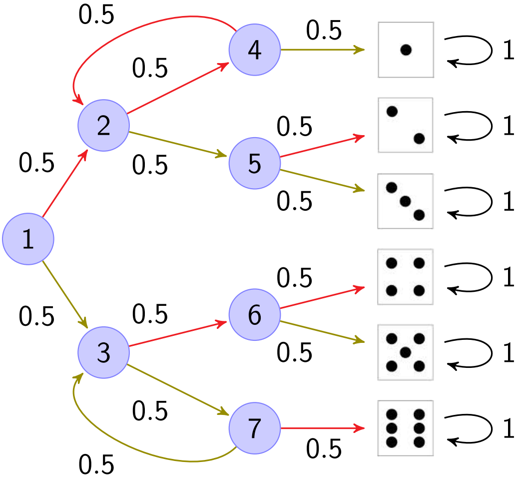

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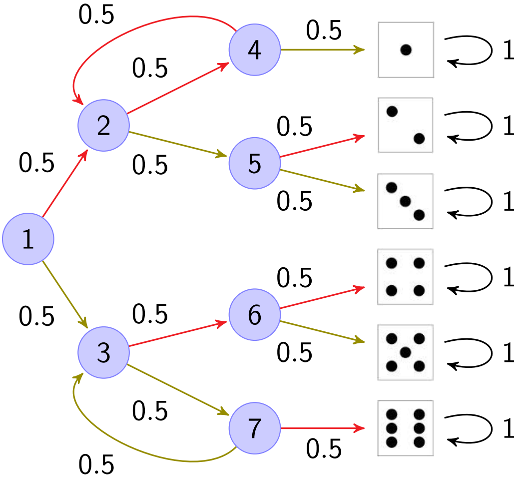
We have to compute an infinite sum!



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Following this approach we compute the reachability property for all the states in the system at the same time!

We let x_s denote the (unknown) value $ProbReach(s, T)$.

We solve the system of linear equations:

$$x_s = \begin{cases} 1 & s \in T \\ 0 & s \text{ cannot reach } T \\ \sum_{s' \in S} P(s, s') \cdot x_{s'} & \text{otherwise} \end{cases}$$

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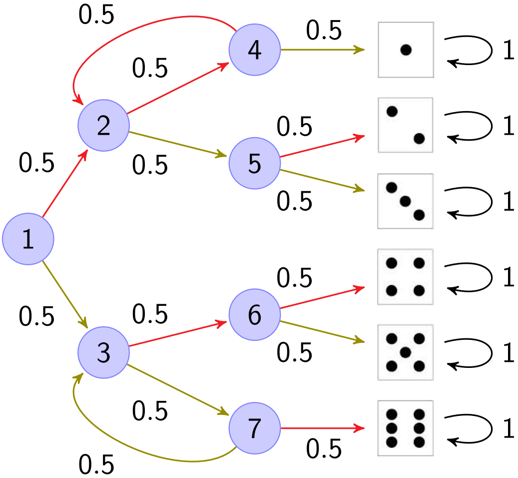

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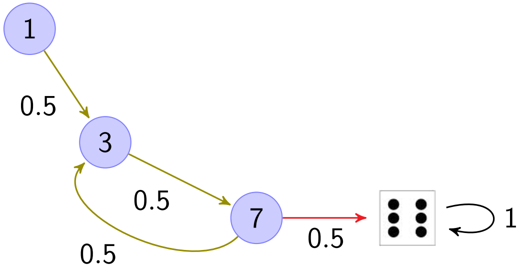

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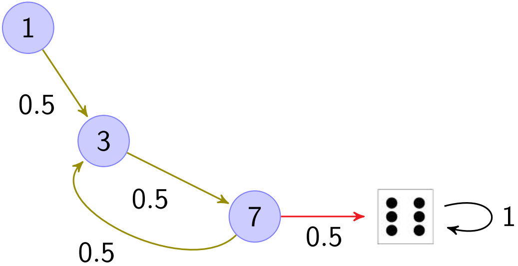
$$
\begin{aligned}
x_1 &= 0.5 \cdot x_3 \\
x_3 &= 0.5 \cdot x_7 \\
x_7 &= 0.5 \cdot x_3 + 0.5 \cdot x_{\text{⚅}} \\
x_{\text{⚅}} &= 1 \\
x_i &= 0
\end{aligned}
$$



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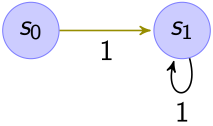

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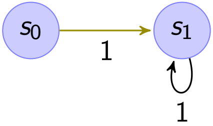

Let us compute the probability to reach $\{s_1\}$. If we do not remove states that are not able to reach s_0 we have the following system:

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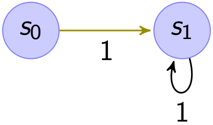

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Any assignment $(x_0, s_1) = (0, p)$ (with $p \in [0, 1]$) is a valid solution!

Bounded Reachability probability

Let (S, P) be a DTMC and $T \subseteq S$, we let

$$ProbReach^{\leq k}(s, T) = Pr(Reach^{\leq k}(s, T))$$

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Function $ProbReach^{\leq k}(s, T)$ can be recursively defined:

$$ProbReach^{\leq k}(s, T) = \begin{cases} 1 & s \in T \\ 0 & k = 0 \ \wedge s \notin T \\ \sum_{s' \in S} P(s, s') \cdot ProbReach^{\leq k-1}(s', T) & \end{cases}$$

To be continued...