Advanced Topics in Software Engineering: Population Models

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Advanced Topics in Software Engineering
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- **R**ecovered.

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Dynamics in *SEIR* model can be described via a CTMC.

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SEIR via CTMC...

Each state of the CTMC has the following form:

$$(x_S, x_E, x_I, x_R)$$

where

- x_S is the number of *suscettibles*;
- x_E is the number of *exposed*;
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If we let $N \in \mathbb{N}$ be the number of citizens in the area, the state space of our CTMC is a subset of $[0, N]^4$.

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In the SEIR system three kinds of events can occur:

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- One **I**nfected becomes **R**ecovered,

Rates of the above events depends on the number/fraction of citizens of the different kinds!

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SEIR via CTMC...

$$(x_S, x_E, x_I, x_R) \xrightarrow{\lambda_{SE}^{(x_S,x_E,x_I,x_R)}} (x_S - 1, x_E + 1, x_I, x_R)$$

$$(x_S, x_E, x_I, x_R) \xrightarrow{\lambda_{EI}^{(x_S,x_E,x_I,x_R)}} (x_S, x_E - 1, x_I + 1, x_R)$$

$$(x_S, x_E, x_I, x_R) \xrightarrow{\lambda_{IR}^{(x_S,x_E,x_I,x_R)}} (x_S, x_E, x_I - 1, x_R + 1)$$

Population models...

A Population Continuous Time Markov Chain (PCTMC) model is a tuple $M = (\mathbf{X}, \mathcal{D}, \mathcal{T}, \mathbf{d}_0)$ where:

- $\mathbf{X} = (X_1, \ldots, X_n)$ is a vector of variables;
- each X_i takes values in a *finite* or *countable* domain $\mathcal{D}_i \subset \mathbb{R}$;
- $\mathcal{D} = \mathcal{D}_0 \times \cdots \times \mathcal{D}_n = \Pi_i \mathcal{D}_i$;
- $\mathbf{d}_0 \in \mathcal{D}$ is the *initial state* of the model;
- $\mathcal{T} = \{\tau_1, \ldots, \tau_m\}$ is the set of *transitions* $\tau_i = (\ell, \mathbf{s}, \mathbf{t}, r)$ where:
  - ℓ is the *label* of the transition;
  - $\mathbf{s} \in \mathbb{R}^n_{\geq 0}$, is the *pre-vector*;
  - $\mathbf{t} \in \mathbb{R}^n_{\geq 0}$, is the *post-vector*;
  - $r : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a *rate function* such that for any $\mathbf{d} \in \mathcal{D}$, if $\mathbf{d} - \mathbf{s} + \mathbf{t} \notin \mathcal{D}$ then $r(\mathbf{d}) = \mathbf{0}$.

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We let $\rightarrow_{\tau_i} \subseteq \mathcal{D} \times \mathbb{R}_{>0} \times \mathcal{D}$ denote the transition relation induced by transition τ_i:

$$\frac{r(\mathbf{d}_1) = \lambda \neq 0 \quad \mathbf{d}_2 = \mathbf{d}_1 - \mathbf{s} + \mathbf{t}}{\mathbf{d}_1 \xrightarrow{\lambda}_{\tau_i} \mathbf{d}_2}$$

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We say that τ_i is *enabled* in $\mathbf{d}_1$ if and only if $r(\mathbf{d}_1) > 0$.

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We say that τ_i is *enabled* in $\mathbf{d}_1$ if and only if $r(\mathbf{d}_1) > 0$.

Finally, function $\rho_{\tau_i} : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$ is used to denote the rate of a transition τ_i from $\mathbf{d}_1$ to $\mathbf{d}_2$:

$$\rho_{\tau_i}(\mathbf{d}_1, \mathbf{d}_2) = \begin{cases} r(\mathbf{d}_1) & \mathbf{d}_2 = \mathbf{d}_1 - \mathbf{s} + \mathbf{t} \\ 0 & \text{otherwise} \end{cases}$$

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Function $\rho_{\mathcal{T}} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ is used to denote the rate of a transition in M from $\mathbf{d}_1$ to $\mathbf{d}_2$:

$$\rho_{\mathcal{T}}(\mathbf{d}_1, \mathbf{d}_2) = \sum_{\tau_i \in \mathcal{T}} \rho_{\tau_i}(\mathbf{d}_1, \mathbf{d}_2)$$

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We let $\rightarrow_{\mathcal{T}} \subseteq \mathcal{D} \times \mathbb{R}_{>0} \times \mathcal{D}$ denote the transition relation induced by transitions $\mathcal{T}$:

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Vector Variables: (S, E, I, R)

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Transitions:

- $\left(\mathsf{S\_E}, \mathbf{1}_{\mathsf{S}}, \mathbf{1}_{\mathsf{E}}, \lambda_e \cdot \frac{\mathsf{X}_S}{N} \cdot \mathsf{X}_I\right)$
- $\left(\mathsf{E\_I}, \mathbf{1}_{\mathsf{E}}, \mathbf{1}_{\mathsf{E}}, \lambda_i \cdot \mathsf{X}_I\right)$
- $\left(\mathsf{I\_R}, \mathbf{1}_{\mathsf{I}}, \mathbf{1}_{\mathsf{R}}, \lambda_r \cdot \mathsf{X}_I\right)$

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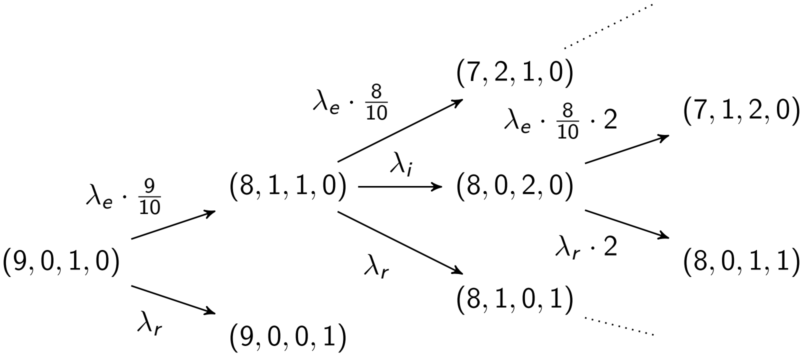
Trajectories. . .

$$\left(\mathsf{S\_E}, \mathbf{1}_{\mathsf{S}}, \mathbf{1}_{\mathsf{E}}, \lambda_e \cdot \frac{X_S}{N} \cdot X_I\right) \quad \left(\mathsf{E\_I}, \mathbf{1}_{\mathsf{E}}, \mathbf{1}_{\mathsf{E}}, \lambda_i \cdot X_I\right) \quad \left(\mathsf{I\_R}, \mathbf{1}_{\mathsf{I}}, \mathbf{1}_{\mathsf{R}}, \lambda_r \cdot X_I\right)$$



From Population Models to CTMC. . .

Let $M = (\mathbf{X}, \mathcal{D}, \mathcal{T}, \mathbf{d}_0)$, $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}$ be a population model, we can easily define the associated CTMC.

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This is obtained as $(\mathcal{D}, \mathbf{R}_M)$ where the rate transition matrix $\mathbf{R}_M$ is defined as follows:

$$\mathbf{R}_M(\mathbf{d}_1, \mathbf{d}_2) = \rho_{\mathcal{T}}(\mathbf{d}_1, \mathbf{d}_2)$$

To be continued...