2. Lexical Analysis

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ToC

Lexical Analysis

```
if (i==j)
   z=0;
else
   z=1;
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Token, Pattern Lexeme

Token

A token is a pair consisting of a token name and an optional attribute value. The token names are the input symbols that the parser processes.

Pattern

A pattern is a description of the form that the lexemes of a token may take. In the case of a keyword as a token, the pattern is just the sequence of characters that form the keyword.

Lexeme

A lexeme is a sequence of characters in the source program that matches the pattern for a token and is identified by the lexical analyzer as an instance of that token.

Lexical Analysis

- Token Class (or Class)
  - In English: *Noun, Verb, Adjective, Adverb, Article, . . .*
  - In a programming language: *Identifier, Keywords, "(", ")", Numbers, . . .*

Lexical Analysis

- Token classes corresponds to sets of strings
- Identifier
  - strings of letter or digits starting with a letter
- Integer
  - a non-empty string of digits
- Keyword
  - "else", "if", "while", . . .
- Whitespace
  - a non-empty sequence of blanks, newlines, and tabs

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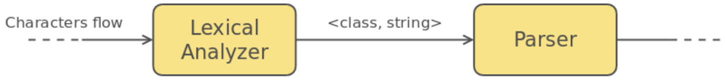
Lexical Analysis

Therefore the role of the lexical analyzer (Lexer) is:

- Classify program substring according to role (token class)
- communicate tokens to parser



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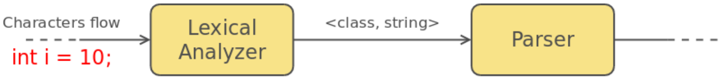

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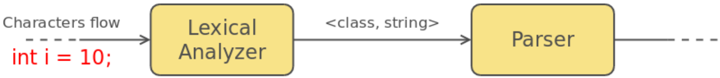

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Lexical Analysis

Let's analyze these lines of code:

```
\tif (i==j)\n\t\tz=0;\n\telse\n\t\tz=1;
```

```
x=0;\n\twhile (x<10) {\n\tx++;\n}
```

Token Classes: Identifier, Integer, Keyword, Whitespace

Lexical Analysis

Therefore an implementation of a lexical analyzer must do two things:

- Recognize substrings corresponding to tokens
  - the lexemes
- Identify the token class for each lexemes

Lexical Analysis - Tricky problems

- FORTRAN rule: whitespace is insignificant
  - i.e. VA R1 is the same as VAR1

```
DO 5 I = 1,25

DO 5 I = 1.25
```

In FORTRAN the "5" refers to a label you will find in the following of the program code

Lexical Analysis - Tricky problems

1. The goal is to partition the string. This is implemented by reading left-to-right, recognizing one token at a time
2. “Lookahead” may be required to decide where one token ends and the next token begins

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if (i==j)
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Lexical Analysis - Tricky problems

- PL/1 keywords are not reserved

```
IF ELSE THEN THEN = ELSE; ELSE ELSE = THEN
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DECLARE(ARG1,...,ARGN)
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Is `DECLARE` a keyword or an array reference?

Need for an unbounded lookahead

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Lexical Analysis - Tricky problems

- C++ template syntax:

```
Foo<Bar>
```

- C++ stream syntax:

```
cin >> var;
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Foo<Bar<Barr>>
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ToC

Languages

Language

Let Σ be a set of characters generally referred as the *alphabet*. A language over Σ is a set of strings of characters drawn from Σ

Alphabet = English character $\Longrightarrow$ Language = English sentences
Alphabet = ASCII $\Longrightarrow$ Language = C programs

Given $\Sigma = \{a, b\}$ examples of simple languages are:

- $\mathcal{L}_1 = \{a, ab, aa\}$
- $\mathcal{L}_2 = \{b, ab, aabb\}$
- $\mathcal{L}_3 = \{\text{s}|\text{s has an equal number of } a \text{ and } b\}$
- ...

Grammar Definition

Grammar

A Grammar is given by a tuple $\mathcal{G} = \langle \mathcal{V}_\mathcal{T}, \mathcal{V}_\mathcal{N}, \mathcal{S}, \mathcal{P} \rangle$ where:

- $\mathcal{V}_\mathcal{T}$: finite and non empty set of terminal symbols (alphabet)
- $\mathcal{V}_\mathcal{N}$: finite set of non terminal symbols s.t. $\mathcal{V}_\mathcal{N} \cap \mathcal{V}_\mathcal{T} = \varnothing$
- $\mathcal{S}$: start symbol of the grammar s.t. $\mathcal{S} \in \mathcal{V}_\mathcal{N}$
- $\mathcal{P}$: is the set of productions s.t. $\mathcal{P} \subseteq (\mathcal{V}^* \cdot \mathcal{V}_\mathcal{N} \cdot \mathcal{V}^*) \times \mathcal{V}^*$ where $\mathcal{V}^* = \mathcal{V}_\mathcal{T} \cup \mathcal{V}_\mathcal{N}$

Derivations

Derivations

Given a grammar $\mathcal{G} = \langle \mathcal{V}_\mathcal{T}, \mathcal{V}_\mathcal{N}, \mathcal{S}, \mathcal{P} \rangle$ a derivation is a sequence of strings $\phi_1, \phi_2, ..., \phi_n$ s.t.
$\forall i \in [1, .., n].\phi_i \in \mathcal{V}^* \wedge \forall i \in [1, ..., n-1].\exists p \in \mathcal{P}.\phi_i \rightarrow^p \phi_{i+1}$.
We generally write $\phi_1 \rightarrow^* \phi_n$ to indicate that from ϕ_1 it is possible to derive ϕ_n repeatedly applying productions in $\mathcal{P}$

Generated Language

The language generated by a grammar $\mathcal{G} = \langle \mathcal{V}_\mathcal{T}, \mathcal{V}_\mathcal{N}, \mathcal{S}, \mathcal{P} \rangle$ corresponds to: $\mathcal{L}(\mathcal{G}) = \{\mathrm{x} | \mathrm{x} \in \mathcal{V}_T^* \wedge \mathcal{S} \rightarrow^* \mathrm{x}\}$

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Chomsky Hierarchy

A hierarchy of grammars can be defined imposing constraints on the structure of the productions in set $\mathcal{P}$ ($\alpha, \beta, \gamma \in \mathcal{V}^*$, $a \in \mathcal{V}_T$, $A, B \in \mathcal{V}_N$):

T0. Unrestricted Grammars:
- Production Schema: *no constraints*
- Recognizing Automaton: Turing Machines

T1. Context Sensitive Grammars:
- Production Schema: $\alpha A \beta \to \alpha \gamma \beta$
- Recognizing Automaton: Linear Bound Automaton (LBA)

T2. Context-Free Grammars:
- Production Schema: $A \to \gamma$
- Recognizing Automaton: Non-deterministic Push-down Automaton

T3. Regular Grammars:
- Production Schema: $A \to a$ or $A \to aB$
- Recognizing Automaton: Finite State Automaton

Meaning function $\mathscr{L}$

Meaning Function

Once you defined a way to describe the strings in a language it is important to define a meaning function L that maps syntax to semantics

- e.g. the case for numbers

- Why using a meaning function?
  - Makes clear what is syntax, what is semantics
  - Allows us to consider notation as a separate issue
  - Because expressions and meanings are not 1 to 1
    - consider the case of arabic number and roman numbers

Warning

It should never happen that the same syntactical structure has more meanings

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Languages

We need to define which is the set of strings in any token class. Therefore we need to choose the right mechanisms to describe such sets:

- Reducing at minimum the complexity needed to recognize lexemes
- Identifying effective and simple ways to describe the patterns

- Regular languages seem to be enough powerful to define all the lexemes in any token class
- Regular expressions are a suitable way to syntactically identify strings belonging to a regular language

Strings

Parts of a string

Terms related to stings:

- prefix of a string *s* is string obtained removing one or more characters from the end of a string *s*
- suffix of a string *s* is string obtained removing one or more characters from the beginning of a string *s*
- substring of a string *s* is obtained deleting any prefix and any suffix from *s*
- proper prefixes, suffixes, and substrings of a string *s* are those, prefixes, suffixes, and substrings, respectively, of *s* that are not ϵ or not equal to *s* itself
- subsequence is any string formed by deleting zero or more not necessarily consecutive positions of *s*

Regular expressions

- Single character: 'c' is a regexp for each $c \in \Sigma$
- Epsilon: ϵ is a regexp
- Union: a+b is a regexp if a and b are regexp (also a|b)
- Concatenation: a · b is a regexp if a and b are regexp (also ab)
- Iteration: a^* is a regexp if a is a regexp

- Algebraic laws for RE:
  - + is commutative and associative
  - concatenation is associative
  - concatenation distributes over +
  - ϵ is the identity for concatenation
  - ϵ is guaranteed in a closure
  - the Kleene star is idempotent

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Meaning function $\mathscr{L}$

- The meaning function $\mathscr{L}$ maps syntax to semantics

$\mathscr{L}(e) = \mathscr{M}$ where e is a RE and $\mathscr{M}$ is a set of strings

Therefore given an alphabet Σ and regular expressions A and B over Σ:

- $\mathscr{L}(\epsilon) = \{\text{“ ”}\}$
- $\mathscr{L}('c') = \{\text{“}c\text{”}\}$ where $c \in \Sigma$
- $\mathscr{L}(A + B) = \mathscr{L}(A) \cup \mathscr{L}(B)$
- $\mathscr{L}(AB) = \{ab | a \in \mathscr{L}(A) \wedge b \in \mathscr{L}(B)\}$
- $\mathscr{L}(A^*) = \{\cup_{i \geq 0} \mathscr{L}(A^i)\}$

RegExp characterization

The regular expressions over Σ are the smallest set including ϵ, all the character 'c' in Σ and that is closed with respect to union, concatenation and iteration.

Regular expressions (syntax)
specify regular languages (semantics)

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Regular expressions (syntax)
specify regular languages (semantics)

Exercise

Consider $\Sigma = \{0, 1\}$. What are the sets defined by the following REs?

- 1^*
- $(1 + 0)1$
- $0^* + 1^*$
- $(0 + 1)^*$

Exercise

Given the regular language identified by $(0 + 1)^*1(0 + 1)^*$ which are the regular expressions identifying the same language among the following one:

- $(01 + 11)^*(0 + 1)^*$
- $(0 + 1)^*(10 + 11 + 1)(0 + 1)^*$
- $(1 + 0)^*1(1 + 0)^*$
- $(0 + 1)^*(0 + 1)(0 + 1)^*$

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Exercise

Choose the regular languages that are correct specifications of the following English-language description:

Twelve-hour times of the form “04:13PM”. Minutes should always be a two digit number, but hours may be a single digit

- $(0+1)?[0-9] : [0-5][0-9](AM+PM)$
- $((0+\epsilon)[0-9]+1[0-2]) : [0-5][0-9](AM+PM)$
- $(0^{*}[0-9]+1[0-2]) : [0-5][0-9](AM+PM)$
- $(0?[0-9]+1(0+1+2) : [0-5][0-9](a+P)M$

Exercise

Describe the languages denoted by the following RegExp:

- $a(a|b)^* a$
- $a^* ba^* ba^* ba^*$
- $((\epsilon|a)b^*)^*$

Regular definitions

For notational convention we give names to certain regular expressions. A regular definition, on the alphabet Σ is sequence of definition of the form:

- $d_1 \rightarrow r_1$
- $d_2 \rightarrow r_2$
- $\ldots$
- $d_n \rightarrow r_n$

where:

- Each d_i is a new symbol, not in Σ and not the same as any other of the d's
- Each r_i is a regular expression over the alphabet $\Sigma \cup \{d_1, d_2, \ldots, d_{i-1}\}$

Using regular definitions

So token of a language can be defined as:

- $letter \rightarrow a|b|...|z|A|B|...|Z$
- $letter\_ \rightarrow letter|\_$
  - compact syntax: $[a - zA - B]$
- $digit \rightarrow 0|1|...|9$
  - compact syntax: $[0 - 9]$
- $Integers \rightarrow (-|\epsilon)digit \cdot digit^*$
- $Identifier \rightarrow letter\_(letter\_|digit)*$
- $ExpNot \rightarrow digit(.digit^+E(+|-)digit^+)?$ (*Exponential Notation*)

Lexical Specification

- At least one: $A^+ \equiv AA^*$
- Union: $A|B \equiv A + B$
- Option: $A? \equiv A + \epsilon$
- Range: $'a' +' b' + ... +' z' \equiv [a - z]$
- Excluded range: complement of $[a - z] \equiv [^\wedge a - z]$

Properties of Regular Languages

Regular languages are closed with respect to union, intersection, complement

Exercise

Write regulare definitions for the following languages:

- All strings of lowercase letters that contains the five wovels in order
- All strings of lowercase letters in which the letters are in ascending lexicographic order
- All strings of digits with no repeated digits
- All strings with an even number of a's and and an odd number of b's

Lexical Specification

We want to derive a regular expression for all tokens of a language:

$s \in \mathscr{L}(R)$ – where R is the RegExp resulting from the sum of the RegExp for all the different kinds ot token

How can we define it?

1. write a regexp for the lexemes of each token class (number, keyword, identifier,. . .)
2. Constructs R matching all lexemes for all tokens
3. Let input be $x_1 \ldots x_n$
   For $1 \le i \le n$ check if $x_1 \ldots x_i \in \mathscr{L}(R_j)$ for some j
4. if success then we know that $x_1 \ldots x_i \in \mathscr{L}(R_j)$ for some j
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LA matching rules

Suppose that at the same time for $i < j$:

- $x_1 \ldots x_i \in \mathscr{L}(R)$
- $x_1 \ldots x_i \ldots x_j \in \mathscr{L}(R)$ or $x_1 \ldots x_i \ldots x_j \in \mathscr{L}(R')$

Which is the match to consider?

longest match rule

Suppose that at the same time for $i \neq j \in [1..n]$ and $R = R_1|R_2|...|R_n$:

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- $x_1 \ldots x_i \ldots x_j \in \mathscr{L}(R)$ or $x_1 \ldots x_i \ldots x_j \in \mathscr{L}(R')$

Which is the match to consider?

longest match rule

Suppose that at the same time for $i \neq j \in [1..n]$ and $R = R_1|R_2|...|R_n$:

- $x_1...x_k \in \mathscr{L}(R_i)$
- $x_1...x_k \in \mathscr{L}(R_j)$

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first one listed rule

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Finite Automata

- Regular Expressions = specification of tokens
- Finite Automata = recognition of tokens

Finite Automaton

A Finite Automaton $\mathcal{A}$ is a tuple $\langle \mathcal{S}, \Sigma, \delta, s_0, \mathcal{F} \rangle$ where:

- $\mathcal{S}$ represents the set of states
- Σ represents a set of symbols (alphabet)
- δ represents the transition function ($\delta : \mathcal{S} \times \Sigma \to \ldots$)
- s_0 represents the start state ($s_0 \in \mathcal{S}$)
- $\mathcal{F}$ represents the set of accepting states ($\mathcal{F} \subseteq \mathcal{S}$)

In two flawors: Deterministic Finite Automata (DFA) and Non-Deterministic Finite Automata (NDFA)

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Acceptance of Strings for Finite Automaton

Derivations

A DFA goes from state s_i to state s_{i+1} consuming from the input the character a if $s_{i+1} = \delta(s_i, a)$. A DFA can go from state s_i to s_j consuming the string $a = a_1 a_2 ... a_n$ if there is a sequence of states $s_{i+1}, ..., s_{i+n-1}$ and $s_j = s_{i+n}$ s.t.
$\forall k \in [1..n].s_{i+k} = \delta(s_{i+k-1}, a_k)$, then we write $s_i \rightarrow^a s_j$
Equivalently the extended transition function $\overline{\delta} : \mathcal{S} \times \Sigma^* \rightarrow \mathcal{S}$ is defined, i.e.
$\delta(\delta(...\delta(s_i, a_1)..., a_{n-1}), a_n) = \overline{\delta}(s_i, a) = s_j$

Acceptance of Strings

A DFA accepts a strings a in the alphabet Σ if there is a derivation from s_0 to a state s_i consuming the string a (i.e. $s_0 \rightarrow^a s_i$) and $s_i \in \mathcal{F}$

Accepted Language

The language accepted by a FSA is constituted by all the strings for which there is a derivation ending in a state in $\mathcal{F}$.

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DFA vs. NFA

Depending on the definition of δ we distinguish between:

- Deterministic Finite Automata (DFA) - $\delta : \mathcal{S} \times \Sigma \rightarrow \mathcal{S}$
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The transition relation δ can be represented in a table (transition table)

Overview of the graphical notation circle and edges (arrows)

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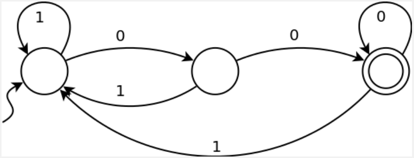
Exercise

Define the following automata:

- DFA for a single 1
- DFA for accepting any number of 1's followed by a single 0
- DFA for any sequence of a or b (possibly empty) followed by 'abb'

Exercise



Which regular expression corresponds to the automaton?

1. (0|1)*
2. (1*|0)(1|0)
3. 1*|(01)*|(001)*|(000*1)*
4. (0|1)*00

ϵ-moves

DFA, NFA and ϵ-moves

- DFA
  - one transition per input per state
  - no ϵ-moves
  - faster
- NFA
  - can have multiple transitions for one input in a given state
  - can have ϵ-moves
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From regexp to NFA

Equivalent NFA for a regexp

The Thompson's algorithm permits to automatically derive a NFA from the specification of a regexp. It defines basic NFA for the basic regexp and rules to compose them:

1. for ϵ
2. for 'a'
3. for AB
4. for A|B
5. for A*

Now consider the regexp for (1|0)*1

NFA to DFA

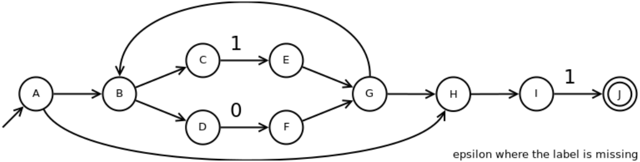
NFA 2 DFA

Given a NFA accepting a language $\mathscr{L}$ there exists a DFA accepting the same language

The derivation of a DFA from an NFA is based on the concept of $\epsilon - closure$. The algorithm to make the transformation is based on:

- $\epsilon - closure(s)$ with $s \in \mathscr{S}$
- $\epsilon - closure(\mathscr{T})$ with $\mathscr{T} \subseteq \mathscr{S}$ i.e. $= \{\cup_{s \in \mathscr{T}} \epsilon - closure(s)\}$
- $move(\mathscr{T}, a)$ with $\mathscr{T} \subseteq \mathscr{S}$ and $a \in \mathscr{L}$

NFA to DFA



epsilon where the label is missing

- build the $\epsilon - closure(\ldots)$ for different states/sets
- build $move(\mathscr{T}, a)$ for different sets and elements

NFA 2 DFA

Subset Construction Algorithm

The Subset constuction algorithm permits to derive a DFA $\langle \mathscr{S}, \Sigma, \delta_D, s_0, \mathscr{F}_D \rangle$ from a NFA $\langle \mathscr{N}, \Sigma, \delta_N, n_0, \mathscr{F}_N \rangle$

$q_0 \leftarrow \epsilon - \text{closure}(\{n_0\})$;
$\mathscr{Q} \leftarrow q_0$;
Worklist $\leftarrow \{q_0\}$;
while (Worklist $\neq \varnothing$) **do**
    take and remove q from Worklist;
    for all ($c \in \Sigma$) **do**
        $t \leftarrow \epsilon - \text{closure}(\text{move}(q, c))$;
        T$[q, c] \leftarrow t$;
        if ($t \notin \mathscr{Q}$) **then**
            $\mathscr{Q} \leftarrow \mathscr{Q} \cup \{t\}$;
            Worklist $\leftarrow$ Worklist $\cup\{t\}$;
        end if
    end for
end while

Simulating DFA and NFA

DFA

$s = s_0$;
$c = nextChar()$;
while (c $\neq$ **eof**) **do**
  $s = move(s, c)$;
  $c = nextChar()$;
end while
if ($s \in \mathscr{F}$) **then return** "yes";
else return "no";
end if

NFA

$S = \epsilon - closure(s_0)$;
$c = nextChar()$;
while (c $\neq$ **eof**) **do**
  $S = \epsilon - closure(move(S, c))$;
  $c = nextChar()$;
end while
if ($S \cap \mathscr{F} \neq \emptyset$) **then return** "yes";
else return "no";
end if

NFA 2 DFA

- Derive a DFA for the regexp: $(a|b)^*abb$
- NFA to DFA for the regexp: $(a|b)^*a(a|b)^{n-1}$

NFA 2 DFA

- Derive a DFA for the regexp: $(a|b)^*abb$
- NFA to DFA for the regexp: $(a|b)^*a(a|b)^{n-1}$

DFA 2 Minimal DFA

Note

Reducing the size of the Automaton does not reduce the number of moves needed to recognize a string, nevertheless it reduces the size of the transition table that could more easily fit the size of a cache

Equivalent states

Two states of a DFA are equivalent if they produce the same "behaviour" on any input string. Formally two states s_i and s_j of a DFA $\mathcal{D} = \langle \mathcal{S}, \Sigma, \delta, q_0, \mathcal{F} \rangle$ are considered equivalent ($s_i \equiv s_j$) **iff** $\forall x \in \Sigma^*.(s_i \rightarrow^x s_i' \wedge s_i' \in \mathcal{F}) \iff (s_j \rightarrow^x s_j' \wedge s_j' \in \mathcal{F})$

DFA 2 Minimal DFA – Hopcroft's Algorithm

Let T a matrix containing information about the equivalence of two states and
let L a matrix containing sets (initially empty) of pairs of states

for all $s_x \in \mathcal{S} \wedge s_y \in \mathcal{S}$ **do**
    $T[s_x, s_y] \leftarrow 0$;   *// All pairs of states are initially marked as equivalent*
end for
for all $s_x \in \mathcal{F} \wedge s_y \in \mathcal{S}/\mathcal{F}$ **do**
    $T[s_x, s_y] \leftarrow 1$; $T[s_y, s_x] \leftarrow 1$;
end for
for all $\langle s_x, s_y \rangle$ s.t. $T[s_x, s_y] = 0 \wedge s_x \neq s_y$ **do**
    if $(\exists c \in \Sigma . T[\delta(s_x, c), \delta(s_y, c)] = 1)$ **then**
        $T[s_x, s_y] \leftarrow 1$; $T[s_y, s_x] \leftarrow 1$;
        for all $\langle s_w, s_z \rangle \in L[s_x, s_y]$ **do**
            $T[s_w, s_z] \leftarrow 1$; $T[s_z, s_w] \leftarrow 1$;
        end for
    else
        for all $c \in \Sigma$ **do**
            if $(\delta(s_x, c) \neq \delta(s_y, c) \wedge (s_x, s_y) \neq (\delta(s_x, c), \delta(s_y, c))$ **then**
                $L[\delta(s_x, c), \delta(s_y, c)] \leftarrow L[\delta(s_x, c), \delta(s_y, c)] \cup \langle s_x, s_y \rangle$;
                $L[\delta(s_y, c), \delta(s_x, c)] \leftarrow L[\delta(s_y, c), \delta(s_x, c)] \cup \langle s_x, s_y \rangle$;
            end if
        end for
    end if
end for

Uniqueness of the minimal DFA

$\exists!$ DFA that recognizes a regular language $\mathscr{L}$ and has minimal number of states

- Minimize DFA for the regexp: $(a|b)^*abb$

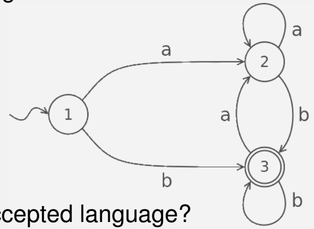
Minimizing Transition Table

The easiest way to represent a DFA is to have a matrix with state and characters. Alternative representations:

- Lists of pairs for each state (character,states)
- hardcoded table into case statements

Example

Consider the following DFA:



Transition Table

states \ characters	a	b
1	2	3
2	2	3
3	2	3

- Which is the accepted language?
- How can the table be represented as a list of pairs?

Exercises

RegExp 2 DFA

- Define an automated strategy to decide if two regular expressions define the same language combininig the algorithms defined so far
- Write a regular expression for all strings of a's and b's which do not contain the substring aab

Regular Languages properties

- Show that the complement of a regular language, on alphabet Σ, is still a regular language
- Show that the intersection of two regular languages, on alphabet Σ, is still a regular language

Scanner issues

Describe the behaviour of a scanner when the two tokens described by the following patterns are considered: ab and $(ab)^*c$. Why a simple scanner is particularly inefficient on a string like 'abababababab'?

Summary

Lexical Analysis

Relevant concepts we have encountered:

- Tokens, Patterns, Lexemes
- Chomsky hierarchy and regular languages
- Regular expressions
- Problems and solutions in matching strings
- DFA and NFA
- Transformations
  - RegExp $\rightarrow$ NFA
  - NFA $\rightarrow$ DFA
  - DFA $\rightarrow$ Minimal DFA