Fundamentals of Reactive Systems

Preliminaries

Francesco Tiezzi

University of Camerino
francesco.tiezzi@unicam.it

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Set Notation

$A \subseteq B$ every element of A is in B

$A \subset B$ if $A \subseteq B$ and there is one element of B not in A

$A \subseteq B$ and $B \subseteq A$ implies $A = B$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ $\quad (\bigcup_{i \in I} A_i)$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ $\quad (\bigcap_{i \in I} A_i)$

$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ *ordered pairs* $\quad (\times_{i=1}^{n} A_i)$

$2^A = \{X \mid X \subseteq A\}$ *powerset*

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Relations

$R \subseteq A \times B$ is a relation on sets A and B $\quad (R \subseteq \times_{i=1}^{n} A_i)$

$$(a, b) \in R \;\equiv\; R(a, b) \;\equiv\; aRb \qquad \textit{notation}$$

$$Id_A = \{(a, a) \mid a \in A\} \qquad \text{(identity)}$$

$$R^{-1} = \{(y, x) \mid (x, y) \in R\} \subseteq B \times A \qquad \text{(inverse)}$$

$$R_1 \cdot R_2 = \{(x, z) \mid \exists y \in B.\ (x, y) \in R_1 \wedge (y, z) \in R_2\} \subseteq A \times C \qquad \text{(composition)}$$

Some basic constructions:

$$\begin{array}{lcl} R^0 & = & Id_A \\ R^{n+1} & = & R \cdot R^n \\ R^* & = & \bigcup_{n \geq 0} R^n \\ R^+ & = & \bigcup_{n \geq 1} R^n \end{array}$$

Note that: $\quad R^1 = R \cdot R^0 = R, \qquad R^* = Id_A \cup R^+ \quad$ and

$R^+ = \{(x, y) \mid \exists n, \exists x_1, \ldots, x_n \text{ with } x_i R x_{i+1}\ (1 \leq i \leq n-1),\ x_1 = x,\ x_n = y\}$

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Properties of Relations

Binary Relations

A binary relation $R \subseteq A \times A$ is (same set A)

reflexive: if $\forall x \in A, (x, x) \in R$,
symmetric: if $\forall x, y \in A, (x, y) \in R \Rightarrow (y, x) \in R$,
antisymmetric: if $\forall x, y \in A, (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$;
transitive: if $\forall x, y, z \in A, (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$

Closure of Relations

$S = R \cup Id_A$ the reflexive closure of R
$S = R \cup R^{-1}$ the symmetric closure of R
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Special Relations

A relation R is

- an **order** if it is reflexive, antisymmetric and transitive
- an **equivalence** if it is reflexive, symmetric and transitive
- a **preorder** if it is reflexive and transitive

Examples

- **orders**: less-than-or-equal-to ($\leqslant$) on $\mathbb{R}$, set inclusion ($\subseteq$),...
- **equivalences**: equal-to ($=$) on $\mathbb{R}$, congruent-mod-n, ...
- **preorders**: reachability in directed graphs, some subtyping,...

Kernel relation

- Given a preorder R its kernel, defined as $K = R \cap R^{-1}$, is an equivalence relation

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Equivalence Classes and Quotient Set

Examples of **equivalence relations**: $R \subseteq A \times A$ (reflexive, symmetric, transitive)

Example: $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid (x \equiv y) \bmod 3\}$

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$[0] = \{0, 3, 6, 9, \ldots\}$

$[1] = \{1, 4, 7, 10, \ldots\}$

$[2] = \{2, 5, 8, 11, \ldots\}$

equivalence classes:

- have a representative

- are disjoint

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An **equivalence class** is a subset C of A such that

$$x, y \in C \quad \Rightarrow \quad (x, y) \in R \qquad \textit{consistent} \qquad \text{and}$$

$$x \in C \wedge (x, y) \in R \quad \Rightarrow \quad y \in C \qquad \textit{saturated}$$

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The **quotient set** Q_A^R of A modulo R is the set of equivalence classes induced by R on A

is a partition of A

Example: $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid (x \equiv y) \bmod 3\}$

$Q_{\mathbb{N}}^R = \{\, [0], [1], [2] \,\}$

Functions

Partial Functions

A *partial function* is a relation $f \subseteq A \times B$ such that

$$\forall x, y, z.\ \ (x, y) \in f \ \wedge\ (x, z) \in f \Rightarrow y = z$$

We denote partial function by $\quad f : A \rightharpoondown B$

Total Functions

A (total) *function* is a partial function $\ f : A \rightharpoondown B\ $ such that

$$\forall x\ \exists y.\ \ (x, y) \in f$$

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Functions (total or partial) can be *monotone*, *continuous*, *injective*, *surjective*, *bijective*, *invertible*...

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Induction Principle

Mathematical Induction

To prove that $P(n)$ holds for every natural number $n \in \mathbb{N}$, prove

1. $P(0)$
2. for any $k \in \mathbb{N}$, $P(k)$ implies $P(k+1)$

Example: Show that $sum(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for every $n \in \mathbb{N}$

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(1) $sum(0) = \frac{0(0+1)}{2} = 0$ *base case*

(2) to show: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ implies $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

assume $sum(n) = \frac{n(n+1)}{2}$, for a generic n

$sum(n+1) = sum(n) + (n+1) =$ *properties of summation*

$= \quad \frac{n(n+1)}{2} + (n+1)$ *inductive hypothesis*

$= \quad \frac{(n+1)(n+2)}{2}$ *qed*

Playful digression

Some “advanced” proof methods

1. Proof by obviousness: So evident it need not to be mentioned
2. Proof by general agreement: All in favor?
3. Proof by majority: When general agreement fails
4. Proof by plausibility: It sounds good
5. Proof by intuition: I have this feeling. . .
6. Proof by lost reference: I saw it somewhere
7. Proof by obscure reference: It appeared in the Annals of Polish Math. Soc. (1854, in polish)
8. Proof by logic: It is on the textbook, hence it must be true
9. Proof by intimidation: Who is saying that it is false!?
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Inductively Defined Sets

basis:	the set I of initial elements of S
induction:	rules R for constructing elements in S from elements in S
closure:	S is the least set containing I and closed w.r.t. R

Natural numbers

$I = \{0\}, \quad R_1:$ if $X \in S$ then $s(X) \in S$

$S = \{0, s(0), s(s(0)), \ldots\}$

$S = Lists(\mathbb{N})$, lists of numbers in $\mathbb{N}$

$I = \{[\,]\}, \quad R_1:$ if $X \in S$ and $n \in \mathbb{N}$ then $[n|X] \in S$

$S = \{[\,], [0], [1], [2], \ldots, [0,0], [0,1], [0,2], \ldots, [1,0], [1,1], [1,2], \ldots\}$

n-ary trees

$I = \{\varepsilon\}, \quad R_1:$ if $X_1, \ldots, X_n \in S$ then $t(X_1, \ldots, X_n) \in S$

$S = \{\varepsilon, t(\varepsilon), t(\varepsilon, \varepsilon), \ldots, t(t(\varepsilon)), \ldots, t(\varepsilon, t(t(\varepsilon), \varepsilon), t(\varepsilon, \varepsilon, \varepsilon)), \ldots\}$

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$I = \{[\,]\}$, $\quad R_1:$ if $X \in S$ and $n \in \mathbb{N}$ then $[n|X] \in S$

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n-ary trees

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Inductively Defined Sets

basis: the set I of initial elements of S
induction: rules R for constructing elements in S from elements in S
closure: S is the least set containing I and closed w.r.t. R

Natural numbers

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Structural Induction

Let us consider a set S inductively defined by a set $C = \{c_1, \ldots, c_n\}$ of constructors of arity $\{a_1, \ldots, a_n\}$ with

- $I = \{c_i(\,) \mid a_i = 0\}$
- R_i : if $X_1, \ldots, X_{a_i} \in S$ then $c_i(X_1, \ldots, X_{a_i}) \in S$

To prove that $P(x)$ holds for every $x \in S$, it is sufficient to prove that

- for every constructor $c_k \in C$ and
- for every $s_1, \ldots, s_k \in S$, where k is the arity of c_k

$$P(s_1), \ldots, P(s_k) \implies P\big(c_k(s_1, \ldots, s_k)\big)$$

Notice that the base case is the one dealing with constructors of arity 0 i.e. with constants

Structural Induction: example

Prove that $\mathit{sum}(\ell) \leq \mathit{max}(\ell) * \mathit{len}(\ell), \quad$ for every $\ell \in \mathit{Lists}(\mathbb{N})$

where

- $\mathit{sum}(\ell)$ is the sum of the elements in the list ℓ
- $\mathit{max}(\ell)$ is the greatest element in ℓ (with $\mathit{max}([\,]) = 0$)
- $\mathit{len}(\ell)$ is the number of elements in ℓ

Structural Induction: example

Exercise: prove $\quad sum(\ell) \leq max(\ell) * len(\ell), \quad$ for every $\ell \in Lists(\mathbb{N})$

$$
\begin{array}{ll}
sum([\,]) = 0 & len([\,]) = 0 \\
sum([n|X]) = n + sum(X) & len([n|X]) = 1 + len(X)
\end{array}
$$

$$
\begin{array}{ll}
max([]) = 0 & \\
max([n|X]) = n & \text{if } max(X) \leq n \\
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(1) $sum([\,]) \leq max([\,]) * len([\,])$

$0 \leq 0 * 0$ *applying definitions*

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prove $\quad sum([n|\ell]) \le max([n|\ell]) * len([n|\ell]) \quad$ for any $n \in \mathbb{N}$

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(a) $n + sum(\ell) \leq n * (1 + len(\ell))$ if $max(\ell) \leq n$ *applying definitions*

$sum(\ell) \quad \leq_{hyp} \quad max(\ell) * len(\ell) \quad \leq_{(a)} \quad n * len(\ell)$ *QED*

Structural Induction: example

Exercise: prove $\mathit{sum}(\ell) \leq \mathit{max}(\ell) * \mathit{len}(\ell)$, for every $\ell \in \mathit{Lists}(\mathbb{N})$

$$\begin{array}{ll} \mathit{sum}([\,]) = 0 & \mathit{len}([\,]) = 0 \\ \mathit{sum}([n|X]) = n + \mathit{sum}(X) & \mathit{len}([n|X]) = 1 + \mathit{len}(X) \end{array}$$

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(b) $n + \mathit{sum}(\ell) \leq \mathit{max}(\ell) + \mathit{max}(\ell) * \mathit{len}(\ell))$ if $n < \mathit{max}(\ell)$ *applying definitions*

$A \leq B$ and $C \leq D$ imply $A + C \leq B + D$ *QED*

Inference Systems

1. I can be written as $\dfrac{}{t}$ (for any $t \in I$)
2. R_i can be written as $\dfrac{p_1 \quad \cdots \quad p_n}{q}$

Meaning: $\vdash t$ and if $\vdash p_1, \ldots, \vdash p_n$ then $\vdash q$

Example: rational numbers $\mathbb{Q}$

$$\frac{}{0 \in N} \qquad \frac{}{1 \in D} \qquad \frac{k \in N}{k+1 \in N} \qquad \frac{k \in D}{k+1 \in D} \qquad \frac{k \in N, \quad h \in D}{k/h \in \mathbb{Q}}$$

A derivation:

$$\frac{\dfrac{\dfrac{}{0 \in N}}{1 \in N} \quad \dfrac{\dfrac{}{1 \in D}}{2 \in D}}{1/2 \in \mathbb{Q}}$$

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Question: why do we need the rules in Red?

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More on Inductively Defined Sets

- $S_{I,R} = \{x \mid \ \vdash x\}$ *the set of all finitely derivable elements*
- $R(X) = \{y \mid \dfrac{x_1 \ \cdots \ x_n}{y} \text{ and } x_1, \ldots x_n \in X\}$ *one step derivation*

X is **closed** under R if $R(X) \subseteq X$ *called a (pre-)fixed point*

R is **monotonic** if $A \subseteq B \;\Rightarrow\; R(A) \subseteq R(B)$

$$\begin{array}{lllll} S^0 & = & R^0(\emptyset) & = & \emptyset \\ S^1 & = & R^1(\emptyset) & = & R(\emptyset) \\ S^2 & = & R^2(\emptyset) & = & R(R(\emptyset)) \\ & \vdots & & & \\ S & \triangleq & \bigcup_{i \in \mathbb{N}} S^i & & \end{array}$$

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Constructing Inductively Defined Sets – an example

$$fib(0) = 0$$
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a sequence of partial functions (under-) approximating *fib*

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Languages

Strings over an alphabet

Let Γ be an alphabet (a finite nonempty set of symbols).
The set $Strings(\Gamma)$ is inductively defined as follows:

- $I = \Gamma \cup \{\varepsilon\}$,
- R_1: if $\quad x, y \in Strings(\Gamma) \quad$ then $\quad xy \in Strings(\Gamma)$
- xy is the concatenation of the strings x and y $\quad (\varepsilon x = x\varepsilon = x)$
- Notation: $\quad \Gamma^* = Strings(\Gamma) \qquad$ (star closure of an alphabet)

An example

$\Gamma = \{a, b\}, \quad Strings(\Gamma) = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$

Languages

- A **language** on Γ is any subset $L \subseteq \Gamma^*$
- They can be defined inductively through formal grammars

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$\Gamma = \{a, b\}, \quad Strings(\Gamma) = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$

Languages

- A **language** on Γ is any subset $L \subseteq \Gamma^*$
- They can be defined inductively through formal grammars

Languages

Strings over an alphabet

Let Γ be an alphabet (a finite nonempty set of symbols).
The set $Strings(\Gamma)$ is inductively defined as follows:

- $I = \Gamma \cup \{\varepsilon\}$,
- R_1: if $\quad x, y \in Strings(\Gamma) \quad$ then $\quad xy \in Strings(\Gamma)$
- xy is the concatenation of the strings x and y $\quad (\varepsilon x = x\varepsilon = x)$
- Notation: $\quad \Gamma^* = Strings(\Gamma) \quad$ (star closure of an alphabet)

An example

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Languages

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Grammars

A **grammar** is a 4-tuple $G = \langle T, NT, S, P \rangle$ where

1. **terminals** T
2. **nonterminals** NT $(T \cap NT = \emptyset)$
3. **start symbol** $S \in NT$
4. **productions** $P \subseteq (T \cup NT)^* \times (T \cup NT)^*$

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$(u, v_1), (u, v_2), \ldots, (u, v_n) \in P$ also written as

$u \to v_1 \mid v_2 \mid \ldots \mid v_n$

or

$u ::= v_1 \mid v_2 \mid \ldots \mid v_n$ *Backus-Naur Normal Form (BNF)*

Grammars – derivation relation

$G = \langle T, N, S, P \rangle$

$$\frac{s = lur \quad t = lvr \quad u \to v}{s \Rightarrow t} \qquad \text{for any production } u \to v \text{ in } P$$

$\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$

Grammars and Languages

The language generated by G is the following set of string of terminal symbols

$$L(G) = \{ w \in T^* \mid S \Rightarrow^* w \}$$

Grammars – example

$T = \{a, b, c\}$ $\quad NT = \{S, B\}$ $\quad$ start symbol: S

$S \rightarrow aBSc \mid abc \qquad Ba \rightarrow aB \qquad Bb \rightarrow bb$

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$$L(G) = \{a^n b^n c^n \mid n \geq 1\}$$

Abstract and Concrete Syntax

When providing the syntax of programming languages we need to worry about precedence of operators or grouping of statements to distinguish, e.g., between:

$$(3+4)*5 \quad \text{and} \quad 3+(4*5),$$

$$\textbf{while } p \textbf{ do } (c_1; c_2) \quad \text{and} \quad (\textbf{while } p \textbf{ do } c_1); c_2$$

Thus, e.g., for arithmetic expressions we have grammars with parenthesis:

$$E \quad ::= \quad n \mid (E) \mid E+E \mid E-E \mid E*E \mid E/E$$

or more elaborate grammars specifying the precedence of operators (like the next one ...)

Abstract and Concrete Syntax

$$
\begin{array}{llll}
E & ::= & E + T \mid E - T \mid T & \text{(expressions)} \\
T & ::= & T * P \mid T/P \mid P & \text{(terms)} \\
P & ::= & N \mid (E) & \text{(atomic expressions)} \\
N & ::= & D\,N \mid D & \text{(numbers)} \\
D & ::= & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 & \text{(digits)}
\end{array}
$$

- When defining the semantics of programming languages, we are only concerned with the meaning of their constructs, not with the theory of how to write programs
- We thus resort to abstract syntax that leaves us the task of adding enough parentheses to programs to ensure they can be built-up in a unique way

Abstract syntax specifies the parse trees of a language; it is the job of concrete syntax to provide enough information through parentheses or precedence rules for a string to parse uniquely

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From Parsing to Execution

Concrete Syntax $\xrightarrow{\textit{defines}}$ $\boxed{\text{Statements}}$ | $2 + 3 * 4$

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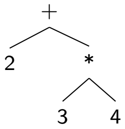

From Parsing to Execution

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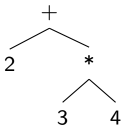
Abstract Syntax	$\xrightarrow{\textit{defines}}$	Syntax Trees		
		$\downarrow$ Execute		$\downarrow$
Semantics	$\xrightarrow{\textit{defines}}$	Meaning of Syntax Trees		14

Labelled Transition Systems

A labelled transition system is a 4-tuple $S = \langle Q, A, \rightarrow, q_0 \rangle$ such that

1. **states** Q
2. **actions** A
3. **transitions** $\rightarrow \subseteq Q \times A \times Q$

   $q \xrightarrow{a} q'$ denotes $(q, a, q') \in \rightarrow$
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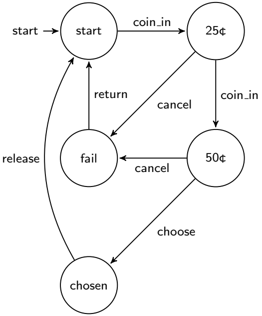
Vending machine:



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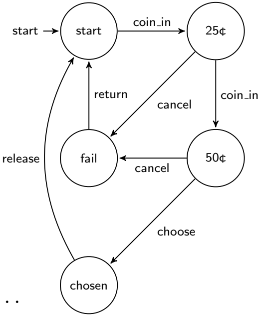

Semantics: traces

$\tau : a_0 a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

τ : coin_in cancel return coin_in coin_in choose release ...

LTS-based Semantics of Arithmetic Expressions

$$\frac{m \circ n = k}{m \circ n \xrightarrow{\circ} k} \quad (\text{op}) \qquad \frac{E_1 \xrightarrow{\circ'} E_1'}{E_1 \circ E_2 \xrightarrow{\circ'} E_1' \circ E_2} \quad (\text{rl}) \qquad \frac{E_2 \xrightarrow{\circ'} E_2'}{E_1 \circ E_2 \xrightarrow{\circ'} E_1 \circ E_2'} \quad (\text{rr})$$

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similarly for $-$ and $/$

Finite State Automata as language recognizers

A *finite state automaton* M is a 5-tuple $M = \langle Q, \Gamma, \rightarrow, q_0, F \rangle$ s.t.

1. **states** Q **finite !**
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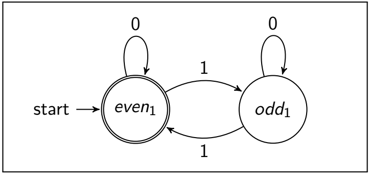
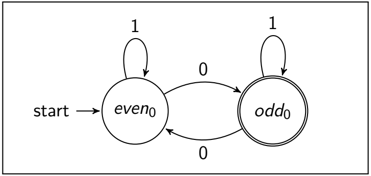
Semantics of Finite State Automata

The language accepted by a Finite State Automata is the set:

$$L(M) = \{ w \in \Gamma^* \mid q_0 \xRightarrow{w} q \text{ and } q \in F \}$$

Some Regular Bit-Strings – $\Gamma = \{0, 1\}$





$L(A_1) = \{w \mid \text{even number of 1's}\}$ $L(A_2) = \{w \mid \text{odd number of 0's}\}$

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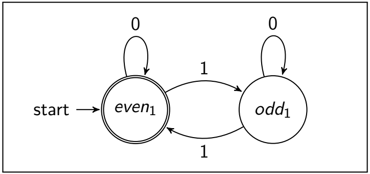

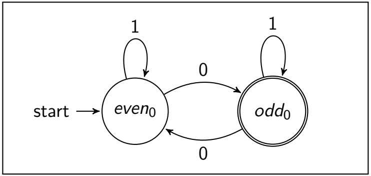

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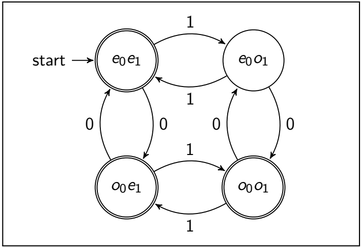

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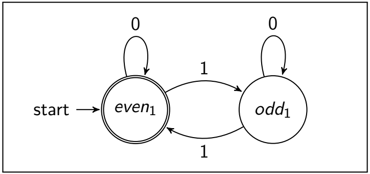

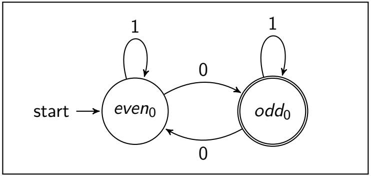

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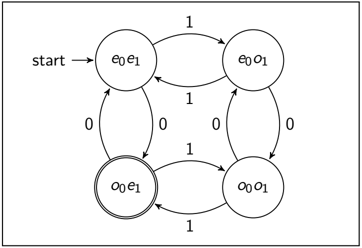

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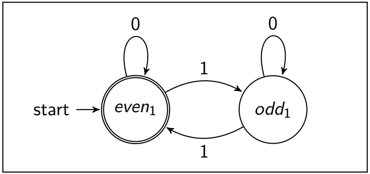

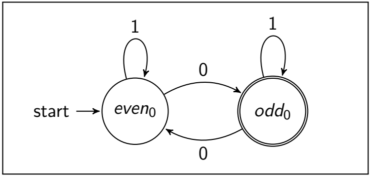

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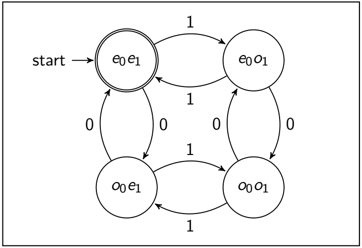

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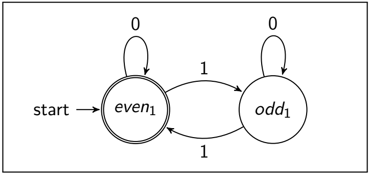

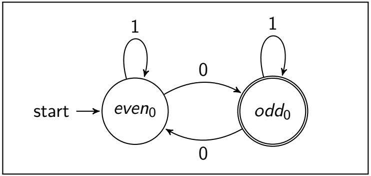

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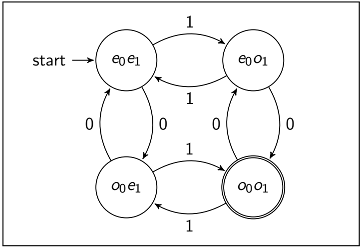

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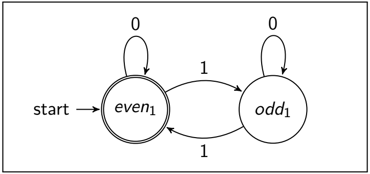

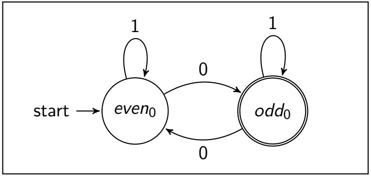

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regular languages are closed w.r.t. the operations of $\cap$, $\cup$, $\setminus$, complement, reversal, concatenation, star closure, . . .

Regular Languages

Chomsky Hierarchy	Grammar Restriction	Language	Abstract Machine
Type 0	unrestricted	recursively enumerable	Turing machines
Type 1	$\alpha A \beta \to \alpha \gamma \beta$	context sensitive	linear bounded automata
Type 2	$A \to \gamma$	context free	nondeterministic pushdown automata
Type 3	$A \to a \quad A \to aB$	regular	finite state automata

with $A,\ B \in NT$, and $a \in T$ and $\alpha,\ \beta,\ \gamma \in (T \cup NT)^*$