Formal Modelling of Software Intensive Systems

Formal Semantics of Regular Expressions

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Formal semantics

Three main approaches to formal semantics of programming languages:

- Operational Semantics (*How a program computes*) [Plotkin, Kahn]:

  Sets of **computations** resulting from the **execution** of programs by an abstract machine

- Denotational Semantics (*What a program computes*) [Strachey, Scott]:

  An input/output **function** that denotes the **effect** of executing the program

- Axiomatic Semantics (*What a program modifies*) [Floyd, Hoare]:

  Pairs of **observable properties** that hold before and after program execution

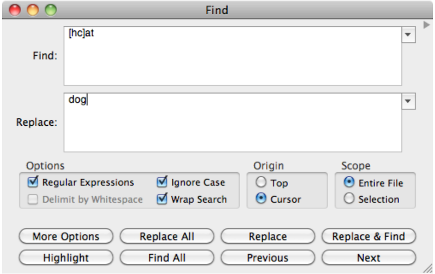
Different purposes, complementary use

A motivating example: regular expressions

Regular expressions

Commonly used for:

- searching and manipulating text based on patterns



Example

Regular expression: `[hc]at` $\Rightarrow$ $(h + c); a; t$

Text: `the cat eats the bat's hat rather than the rat`

Matches: `cat, hat`

A motivating example: regular expressions

Regular expressions

Commonly used for:

- searching and manipulating text based on patterns
- representing regular languages in a compact form
- describing sequences of actions that a system can execute

- Regular expressions as a simple programming language
  - Programming constructs: sequence, choice, iteration, stop
- We define the semantics of regular expressions by applying the three approaches
- We show that the three semantics are consistent

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Regular expressions: syntax and informal semantics

Abstract syntax

$$E \;::=\; 0 \;\mid\; 1 \;\mid\; a \;\mid\; E + E \;\mid\; E ; E \;\mid\; E^*$$

Operators precedence

* binds more than $+$ and ;   ; binds more than $+$

Informal semantics

- 0 is the empty event
- 1 is the terminal event
- a is an event (or atomic action) where $a \in A$, with A finite alphabet
- $E + F$ can be either E or F (choice operator)
- $E ; F$ is the expression E followed by F (sequencing)
- E^* is an n-length sequence of E with $n \geq 0$ (Kleene star)

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Regular expressions: informal semantics

With an informal semantics the meaning of composite expressions may be not clear

Example

$$(a + b)^* \qquad\qquad (a^* + b^*)^*$$

- They are syntactically different
- What about their meaning?

We shall apply the three approaches used for defining formal semantics to regular expressions

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Regular expressions: operational semantics

We introduce an **abstract machine** for **executing** regular expressions

Transition relation

- Is a ternary relation $E \xrightarrow{\mu} F$, where $\mu \in A \cup \{\varepsilon\}$ (ε empty action)
- Is defined by an inference system
- Describes, by induction on the structure of the expressions, the behaviour of a machine that takes as input a regular expression and executes it

For a generic operator *op* we shall have one or more rules like:

$$\frac{E_{i_1} \xrightarrow{\alpha_1} E'_{i_1} \quad \cdots \quad E_{i_m} \xrightarrow{\alpha_m} E'_{i_m}}{op(E_1, \cdots, E_n) \xrightarrow{\alpha} op(E'_1, \cdots, E'_n)} \qquad \text{where } \{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}.$$

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Transition relation rules

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(Atom) $$\frac{}{a \xrightarrow{a} 1} \quad a \in A$$

(Sum$_1$) $$\frac{E \xrightarrow{\mu} E'}{E + F \xrightarrow{\mu} E'}$$

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(Seq$_1$) $$\frac{E \xrightarrow{a} E'}{E; F \xrightarrow{a} E'; F}$$

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Structural Operational Semantics (SOS [Plotkin])

Transition relation is the least relation satisfying the above rules

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1 indicates the terminal state: the machine has completed the execution and loops by executing the empty action

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Expression a executes action a and stops

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$E + F$ can behave either as E or as F: if E evolves to E' by performing action μ then $E + F$ can evolve to E' by performing μ; similarly for F

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$E; F$ executes the actions of E and, afterwards, the actions of F

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No rule for 0: expression 0 does nothing

0 indicates the deadlock state: the machine is stuck

The automaton associated to a regular expression

The SOS inference rules implicitly defines a particular automaton for each regular expression E (essentially a fragment of the whole LTS):

- the initial state is E (we shall often omit to mark it)
- the set of labels is A
- the set of states consists of all regular expressions that can be reached starting from E via a sequence of transitions
- the transition relation is the one induced from the SOS rules
- the only final state is 1 (we shall often omit to mark it)

Semantic correspondence

Given any regular expression E, the automaton generated by the SOS rules has the property of recognizing exactly the language $\mathcal{L}[\![E]\!]$, but it is not the unique automaton satisfying such property.
Other "similar" automata might have less (or more) ε transitions.

A few examples for Regular Expressions

$(a+b)^* \xrightarrow{a} 1; (a+b)^*$

$$\cfrac{\cfrac{\overline{a \xrightarrow{a} 1}\ (Atom)}{a+b \xrightarrow{a} 1}\ (Sum_1)}{(a+b)^* \xrightarrow{a} 1; (a+b)^*}\ (Star_2)$$

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Regular expressions: operational semantics

Definition (Traces of Regular expressions)

- Let E be a regular expression and $s \in A^*$ be a string, we write $E \stackrel{s}{\Rightarrow} E'$ if there exists $\mu_1, \ldots, \mu_n \in A \cup \{\varepsilon\}$ $(n \geq 0)$ s.t.:
  1. the string $\mu_1 \ldots \mu_n$ coincides with s (up to some occurrence of ε)
  2. $E \xrightarrow{\mu_1} E_1 \xrightarrow{\mu_2} E_2 \xrightarrow{\mu_3} \ldots \xrightarrow{\mu_n} E_n \equiv E'$ ($\equiv$ syntactical equiv.)

- The set of *traces* of E is the set of strings

$$\text{Traces}(E) = \{s \in A^* : \ E \stackrel{s}{\Rightarrow} 1\}$$

Definition (Trace equivalence)

Two regular expressions E and F are *trace equivalent* if

$$\text{Traces}(E) = \text{Traces}(F)$$

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$$(a + b)^* \qquad\qquad (a^* + b^*)^*$$

- They are syntactically different
- Are they semantically equivalent?

We have to show that:

- s is a trace of $(a + b)^*$ if and only if s is a trace of $(a^* + b^*)^*$

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if s is a trace of $(a+b)^*$ then s is a trace of $(a^*+b^*)^*$

Induction on the length of s.

- *Base step*: $|s| = 0$ (i.e., $s = \varepsilon$). Trivial: (Star$_1$), $(a^*+b^*)^* \xrightarrow{\varepsilon} 1$
- *Inductive step*: $|s| > 0$, then $s = as'$ or $s = bs'$; w.l.o.g. assume $s = as'$.
  The only possible a-transition for $(a+b)^*$ is $(a+b)^* \stackrel{a}{\Rightarrow} (a+b)^*$
  This is proved via the following derivations:

$$\cfrac{\cfrac{\cfrac{}{a \xrightarrow{a} 1}\,(Atom)}{a+b \xrightarrow{a} 1}\,(Sum_1)}{(a+b)^* \xrightarrow{a} 1;(a+b)^*}\,(Star_2)
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- *Inductive step*: $|s| > 0$, then $s = as'$ or $s = bs'$; w.l.o.g. assume $s = as'$.
  The only possible a-transition for $(a+b)^*$ is $(a+b)^* \stackrel{a}{\Rightarrow} (a+b)^*$
  By hypothesis, $(a+b)^* \stackrel{as'}{\Longrightarrow} 1$, thus $(a+b)^* \stackrel{s'}{\Longrightarrow} 1$.
  By induction, we have $(a^*+b^*)^* \stackrel{s'}{\Longrightarrow} 1$, thus it is sufficient to prove $(a^*+b^*)^* \stackrel{a}{\Rightarrow} (a^*+b^*)^*$ to conclude that $(a^*+b^*)^* \stackrel{s}{\Rightarrow} 1$.

Regular expressions: operational semantics

if s is a trace of $(a+b)^*$ then s is a trace of $(a^*+b^*)^*$

Induction on the length of s.

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  $(a^*+b^*)^* \stackrel{a}{\Rightarrow} (a^*+b^*)^*$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \xrightarrow{a} 1}\,(Atom)}{a^* \xrightarrow{a} 1; a^*}\,(Star_2)}{a^*+b^* \xrightarrow{a} 1; a^*}\,(Sum_1)}{(a^*+b^*)^* \xrightarrow{a} 1; a^*; (a^*+b^*)^*}\,(Star_2)$$

$$\dfrac{\dfrac{}{1 \xrightarrow{\varepsilon} 1}\,(Tic)}{1; a^*; (a^*+b^*)^* \xrightarrow{\varepsilon} a^*; (a^*+b^*)^*}\,(Seq_2)$$

$$\dfrac{\dfrac{}{a^* \xrightarrow{\varepsilon} 1}\,(Star_1)}{a^*; (a^*+b^*)^* \xrightarrow{\varepsilon} (a^*+b^*)^*}\,(Seq_2)$$

Regular expressions: operational semantics

The abstract machine that describes the execution of a regular expression is a *finite state automaton*

Definition (Regular expressions as finite state automata)

Let E be a reg. expr., the finite state automaton associated to E is

$$M_E = \big(Q_E,\ A,\ \rightarrow_E,\ E,\ \{1\}\big)$$

- *States:* $Q_E = \{F \mid \exists s \in A^*.\ E \stackrel{s}{\Rightarrow} F\}$ (expressions from E)
- *Actions:* A (alphabet of E)
- *Transition relation:* $\rightarrow_E$ s.t. $F \xrightarrow{\mu}_E F'$ if $F \xrightarrow{\mu} F'$ with $\mu \in A \cup \{\varepsilon\}$
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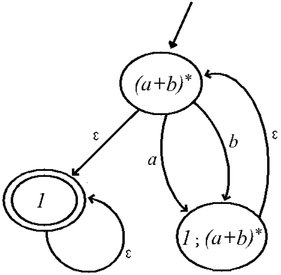
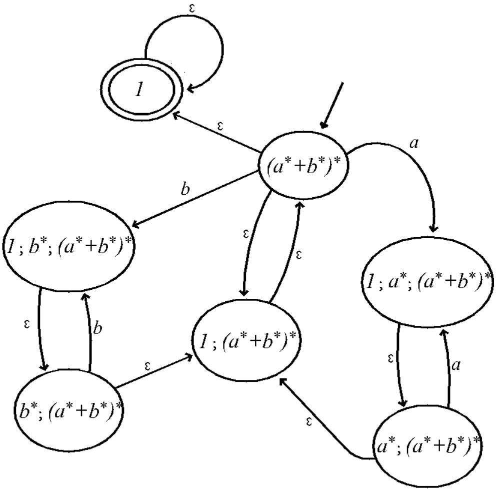
Regular expressions: operational semantics

Automata associated to $(a+b)^*$ and $(a^*+b^*)^*$





Regular expressions: operational semantics

Theorem

Let E be a regular expression and M_E the associated automaton, then

$$\mathrm{Traces}(E) = L(M_E)$$

where $L(M_E) = \{s \in A^* : \ E \stackrel{s}{\Longrightarrow}_E 1\}$ (language accepted by M_E)

Proof (*sketch*). Two cases:

- $\subseteq$ If $w \in \mathrm{Traces}(E)$, then $E \stackrel{w}{\Longrightarrow} 1$. The proof that $w \in L(M_E)$ proceeds by induction on the length of w.
- $\supseteq$ Given $w \in L(M_E)$, we prove by induction on the length of w that $w \in \mathrm{Traces}(E)$.

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Regular expressions: denotational semantics

Denotational Semantics (*What a program computes*)

- an input/output **relation** that denotes the **effect** of executing the program
  - *semantic function*
- associate to each program a mathematical object, called *denotation*, that represents its meaning

Operators on Languages

To define semantics interpretation function for regular expressions, we need some operators on languages. If L, L_1 and L_2 are sets of strings:

- $L_1 \cdot L_2 = \{xy : \ x \in L_1 \ \text{ and } \ y \in L_2\}$
- $L^* \ = \bigcup_{n \geq 0} L^n$ where
  - $L^0 \ = \ \{\varepsilon\}$
  - $L^{n+1} \ = \ L \cdot L^n$

We have: $\emptyset \cdot L = L \cdot \emptyset = \emptyset$ (Why?)

Regular expressions: denotational semantics

Semantic function $\mathcal{L}$ for regular expressions

The denotational semantics is inductively defined by the rules below and associates a subset of A^* to each regular expressions:

$$\mathcal{L}[\![\ ]\!] : R.E. \ \rightarrow \ 2^{A^*}$$

$$\mathcal{L}[\![0]\!] = \emptyset$$

$$\mathcal{L}[\![1]\!] = \{\varepsilon\}$$

$$\mathcal{L}[\![a]\!] = \{a\} \quad (\text{for } a \in A)$$

$$\mathcal{L}[\![E + F]\!] = \mathcal{L}[\![E]\!] \cup \mathcal{L}[\![F]\!]$$

$$\mathcal{L}[\![E\,;F]\!] = \mathcal{L}[\![E]\!] \cdot \mathcal{L}[\![F]\!]$$

$$\mathcal{L}[\![E^*]\!] = (\mathcal{L}[\![E]\!])^*$$

Regular expressions: denotational semantics

Example

$$(a + b)^* \qquad\qquad (a^* + b^*)^*$$

- They are syntactically different
- Are they semantically equivalent?

We have to show that:

- $\mathcal{L}[\![(a + b)^*]\!] \subseteq \mathcal{L}[\![(a^* + b^*)^*]\!]$
- vice versa

Regular expressions: denotational semantics

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Equivalence result

Theorem (operational and denotational semantics are equivalent)

Let E be a regular expression, it holds that:

$$w \in \text{Traces}(E) \iff w \in \mathcal{L}[\![E]\!]$$

Proof. Two cases:

- $\Rightarrow$ By induction on the structure of E.
- $\Leftarrow$ By induction on the structure of E.

Property

Let E and F regular expressions and s a string.

$$E; F \stackrel{s}{\Longrightarrow} 1 \quad \text{implies} \quad \exists x, y \text{ s.t. } s = xy \text{ and } E \stackrel{x}{\Longrightarrow} 1,\ F \stackrel{y}{\Longrightarrow} 1$$

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Regular expressions' semantics: equivalence result

Proof ($\Rightarrow$). By induction on the structure of E.

$E \equiv 0$ Trivial, because $\text{Traces}(0) = \emptyset = \mathcal{L}[\![0]\!]$.

$E \equiv 1$ Trivial, because $\text{Traces}(1) = \{\varepsilon\} = \mathcal{L}[\![1]\!]$.

$E \equiv a$ Trivial, because $\text{Traces}(a) = \{a\} = \mathcal{L}[\![a]\!]$.

$E \equiv E_1 + E_2$ If $w \in \text{Traces}(E_1 + E_2)$, then $\exists\ \mu \in A \cup \{\varepsilon\}$ and $w' \in A^*$ with $w = \mu w'$ and

$$E_1 + E_2 \xrightarrow{\mu} F \xRightarrow{w'} 1$$

where

$$E_1 \xrightarrow{\mu} F \xRightarrow{w'} 1 \qquad \text{or} \qquad E_2 \xrightarrow{\mu} F \xRightarrow{w'} 1$$

By inductive hypothesis

$$w \in \mathcal{L}[\![E_1]\!] \qquad \text{or} \qquad w \in \mathcal{L}[\![E_2]\!]$$

Thus, $w \in \mathcal{L}[\![E_1]\!] \cup \mathcal{L}[\![E_2]\!] = \mathcal{L}[\![E_1 + E_2]\!]$.

Equivalence result

$E \equiv E_1; E_2$ If $w \in \text{Traces}(E_1; E_2)$, by the previous property, $\exists\, x, y$ s.t.

$$E_1 \stackrel{x}{\Longrightarrow} 1 \quad \text{and} \quad E_2 \stackrel{y}{\Longrightarrow} 1$$

with $w = xy$. By inductive hypothesis, we have

$$x \in \mathcal{L}[\![E_1]\!] \quad \text{and} \quad y \in \mathcal{L}[\![E_2]\!],$$

and, hence, $w \in \mathcal{L}[\![E_1]\!] \cdot \mathcal{L}[\![E_2]\!] = \mathcal{L}[\![E_1; E_2]\!]$.

$E \equiv E_1^*$ Let $S(E_1^*, w)$ be the number of application of $(Star_2)$ in $E_1^* \stackrel{w}{\Longrightarrow} 1$.
We demonstrate by induction on $n = S(E_1^*, w)$ that

$$w \in \mathcal{L}^n[\![E_1]\!]. \qquad\qquad (\mathcal{L}^n[\![E_1]\!] \text{ stands for } (\mathcal{L}[\![E_1]\!])^n)$$

...

Equivalence result

$E \equiv E_1^*$ ...

If $S(E_1^*, w) = 0$, no $(Star_2)$ but $(Star_1)$ used, thus $w = \varepsilon$.
By definition, $\varepsilon \in \mathcal{L}^0[\![E_1]\!] = \{\varepsilon\}$.
If $S(E_1^*, w) = n + 1$, then $\exists\, x, y$ s.t. $w = xy$ and

$$E_1^* \stackrel{x}{\Longrightarrow} E_1^* \stackrel{y}{\Longrightarrow} E_1^* \xrightarrow{\varepsilon} 1$$

with $S(E_1^*, x) = n$.
By (local) induction hypothesis $x \in \mathcal{L}^n[\![E_1]\!]$. Since $S(E_1^*, y) = 1$, $(Star_2)$ is applied only once in $E_1^* \stackrel{y}{\Longrightarrow} E_1^*$, thus $\exists\, \mu \in A \cup \{\varepsilon\}$ and $y' \in A^*$ s.t. $y = \mu y'$, $E_1 \xrightarrow{\mu} E'$ and

$$E_1^* \xrightarrow{\mu} E'; E_1^* \stackrel{y'}{\Longrightarrow} E_1^*.$$

Since $E'; E_1^* \stackrel{y'}{\Longrightarrow} E_1^*$ does not use $(Star_2)$, we have $E' \stackrel{y'}{\Longrightarrow} 1$ and, hence, $E_1 \stackrel{\mu y'}{\Longrightarrow} 1$. By (structural) inductive hypotesis, $y \in \mathcal{L}[\![E_1]\!]$. Using $x \in \mathcal{L}^n[\![E_1]\!]$, we conclude.

Equivalence result

Proof ($\Leftarrow$). By induction on the structure of E.

For the sake of simplicity, we only consider the case:

$E \equiv E_1^*$ If $w \in \mathcal{L}[\![E_1^*]\!]$, then $\exists\, n$ s.t. $w \in \mathcal{L}^n[\![E_1]\!]$.

Then, $\exists\, x_1, \ldots, x_n \in \mathcal{L}[\![E_1]\!]$ s.t. $w = x_1 \cdots x_n$.

By inductive hypothesis, $x_i \in \text{Traces}(E_1)$, that is $E_1 \xRightarrow{x_i} 1$.

By repeatedly applying $(Star_2)$, we obtain $E_1^* \xRightarrow{x_i} 1; E_1^*$.

Since $1; E_1^* \xrightarrow{\varepsilon} E_1^*$, by (Seq_2), and $E_1^* \xrightarrow{\varepsilon} 1$, by$(Star_1)$, we have

$$E_1^* \xRightarrow{x_1} 1; E_1^* \xRightarrow{x_2} 1; E_1^* \;\cdots\; \xRightarrow{x_n} 1; E_1^* \xrightarrow{\varepsilon} 1$$

and, therefore, $E_1^* \xRightarrow{w} 1$.

Regular expressions: axiomatic semantics

Axiomatic Semantics (*What a program modifies*)

- it relates **observable properties** before and after program execution
  - in stateful languages, e.g., if the initial state of a program fulfils the precondition and the program terminates, then the final state is guaranteed to fulfil the postcondition
- it consists of a set of axioms and inference rules that define a **relation**

Axiomatic semantics of regular expressions

- no state in regular expressions
- the observed property is the capability of equivalent expressions to represent the same regular language
- axioms and rules define an equivalence relation $E = F$ that partition the set of all expressions

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Regular expressions: axiomatic semantics

Axioms for $E = F$

$$
\left.\begin{array}{ll}
E + (F + G) = (E + F) + G & \text{(assoc +)} \\
E + F = F + E & \text{(comm +)} \\
E + 0 = E & \text{(unit +)}
\end{array}\right\} \text{(monoid+)}
$$

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\left.\begin{array}{ll}
E\,;(F\,;G) = (E\,;F)\,;G & \text{(assoc ;)} \\
1\,;E = E & \text{(unit ;)}
\end{array}\right\} \text{(monoid ;)}
$$

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\left.\begin{array}{ll}
E\,;(F + G) = E\,;F + E\,;G & \text{(distribL)} \\
(E + F)\,;G = E\,;G + F\,;G & \text{(distribR)} \\
0\,;E = 0 & \text{(absorb 0)}
\end{array}\right\} \text{(modulo +, ;)}
$$

$$
E + E = E \quad \} \ \text{(idemp +)}
$$

$$
\left.\begin{array}{ll}
E^* = 1 + E^*\,;E & \text{(unfolding)} \\
E^* = (1 + E)^* & \text{(absorb *)} \\
0^* = 1 & (0^0)
\end{array}\right\} \text{(rules *)}
$$

Regular expressions: axiomatic semantics

Rules for $E = F$

Rule 1 (Substitution):

$$\frac{E = F \quad G = H}{G' = H \quad G' = G}$$

where G' is obtained from G by replacing an occurrence of E by F

Rule 2 (Equation solution):

$$\frac{E = E\,;F + G}{E = G\,;F^*}$$

if F does not produce ε

Regular expressions: axiomatic semantics

- The axioms are sound w.r.t. the observed property, i.e. $=$ equates expressions representing the same language
  - E.g., given $0\,;E = 0$, we have:

$$\mathcal{L}[\![0\,;E]\!] = \mathcal{L}[\![0]\!] \cdot \mathcal{L}[\![E]\!] = \emptyset \cdot \mathcal{L}[\![E]\!] = \emptyset = \mathcal{L}[\![0]\!]$$

- Applying the axiomatic approach could be more laborious
  - E.g., proving $E\,;0 = 0$ requires the following inference:

$$\dfrac{\dfrac{}{0\,;0^* = 0}\,(\mathit{absorb}\ 0) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{0 = 0\,;0}\,(\mathit{absorb}\ 0) \quad E\,;0 = E\,;0}{E\,;0\,;0 = E\,;0}\,(\mathit{rule}\ 1) \quad \dfrac{}{E\,;0+0 = E\,;0}\,(\mathit{unit}\ +)}{E\,;0\,;0+0 = E\,;0}\,(\mathit{rule}\ 1)}{E\,;0 = 0\,;0^*}\,(\mathit{rule}\ 2)}{E\,;0 = 0}\,(\mathit{rule}\ 1)$$

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Regular expressions' semantics: equivalence result

Theorem (axiomatic and denotational semantics are equivalent)

Let E and F be regular expressions, it holds that:

$$E = F \iff \mathcal{L}[\![E]\!] = \mathcal{L}[\![F]\!]$$

Proof (sketch). Two cases:

- $\Rightarrow$ *(Soundness)* Easy to prove
- $\Leftarrow$ *(Completeness)* Require a bit of work (e.g., expression normalization)

Corollary

The three semantics for regular expressions are equivalent

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