

Model Checking I

alias

Reactive Systems Verification

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Topics

- Recall of Regular Expressions.
- Omega-Regular Expressions.
- Non-deterministic Büchi Automata (NBA).
- NBAs for Linear Time Properties.
- From NBAs to Omega-Regular Expressions and Back.
- Closure properties of Omega-Regular properties.
- Deterministic Büchi Automata (DBA) and their expressive power.
- Generalized Non-deterministic Büchi Automata (GNBA).
- From GNBA to NBAs.

Material

Reading:

Chapter 4 of the book, pages 151–198.

More:

The slides in the following pages are taken from the material of the course “Introduction to Model Checking” held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

regular safety properties

ω -regular properties



model checking with Büchi automata

Linear Temporal Logic

Computation-Tree Logic

Equivalences and Abstraction

idea: define **regular LT properties** to be those languages of **infinite words** over the alphabet 2^{AP} that have a representation by a **finite automata**

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 - NFA**-representation for the **bad prefixes**
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 - * **ω -automata**, i.e., acceptors for infinite words
 - * **ω -regular expressions**

remind: syntax and semantics of regular expressions
over some alphabet $\Sigma = \{A, B, \dots\}$

$$\alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 \cdot \alpha_2 \mid \alpha^*$$

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$$\mathcal{L}(\emptyset) = \emptyset$$

$$\mathcal{L}(\epsilon) = \{\epsilon\}$$

$$\mathcal{L}(A) = \{A\}$$

$$\mathcal{L}(\alpha_1 + \alpha_2) = \mathcal{L}(\alpha_1) \cup \mathcal{L}(\alpha_2) \quad \text{union}$$

$$\mathcal{L}(\alpha_1 \cdot \alpha_2) = \mathcal{L}(\alpha_1) \mathcal{L}(\alpha_2) \quad \text{concatenation}$$

$$\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^* \quad \text{Kleene closure}$$

regular expressions:

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regular expressions + ω -operator α^ω

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for $L \subseteq \Sigma^*$:

$$L^\omega \stackrel{\text{def}}{=} \{w_1 w_2 w_3 \dots : w_i \in L \text{ for all } i \geq 1\}$$

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note: $L^\omega \subseteq \Sigma^\omega$ if $\epsilon \notin L$

syntax of ω -regular expressions over alphabet Σ :

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega \quad \text{where}$$

α_i, β_i are regular expressions over Σ s.t. $\varepsilon \notin \mathcal{L}(\beta_i)$

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- language of $(A^* \cdot B)^\omega + (B^* \cdot A)^\omega$

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A language $L \subseteq \Sigma^\omega$ is called ω -regular iff there exists an ω -regular expression γ s.t.

$$L = \mathcal{L}_\omega(\gamma)$$

alphabet $\Sigma = \{A, B\}$

- set of all infinite words over Σ containing only finitely many A 's

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$$(B^* . A^+ . B)^* . B^\omega + (B^* . A^+ . B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha . \alpha^*$.

Provide an ω -regular expression for ...

alphabet $\Sigma = \{A, B\}$

- set of all infinite words over Σ containing only finitely many A 's

$$(A + B)^* . B^\omega$$

- set of all infinite words where each A is followed immediately by letter B

$$(B^* . A . B)^* . B^\omega + (B^* . A . B)^\omega$$

- set of all infinite words where each A is followed eventually by letter B

$$(B^* . A^+ . B)^* . B^\omega + (B^* . A^+ . B)^\omega \equiv (A^* . B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha . \alpha^*$.

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Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \vee \neg b$

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Let Φ be an invariant condition and let

$$\{A \subseteq AP : A \models \Phi\} = \{A_1, \dots, A_k\}$$

Then: invariant “always Φ ” $\hat{=} (A_1 + \dots + A_k)^\omega$

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Indeed: each invariant is ω -regular

- “infinitely often a ”

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$$((\emptyset + \{b\})^* \cdot (\{a\} + \{a, b\}))^\omega$$

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Examples for $AP = \{a, b\}$:

- “always a ” (or any other invariant)
- “infinitely often a ”
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$$(2^{AP})^* \cdot (\{a\} + \{a, b\}) \cdot (2^{AP})^\omega$$

where $2^{AP} \cong \emptyset + \{a\} + \{b\} + \{a, b\}$

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symbolic notation for ω -regular properties

... using **formulas** instead of **sums**

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- “infinitely often a ”

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- “whenever a then b will hold somewhen later”

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \vee \neg b$

$$(a \vee \neg b)^\omega \cong (\emptyset + \{a\} + \{a, b\})^\omega$$

- “infinitely often a ”

$$((\neg a)^*.a)^\omega \cong ((\emptyset + \{b\})^* . (\{a\} + \{a, b\}))^\omega$$

- “from some moment on a ”:

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- “whenever a then b will hold somewhen later”

$$((\neg a)^*.a.true^*.b)^* . (\neg a)^\omega + ((\neg a)^*.a.true^*.b)^\omega$$

syntax as for **NFA**



nondeterministic finite automata

syntax as for **NFA**



nondeterministic finite automata

semantics: language of infinite words

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

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- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states

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- $F \subseteq Q$ set of **final states**, also called **accept states**

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run for a word $A_0 A_1 A_2 \dots \in \Sigma^\omega$:

state sequence $\pi = q_0 q_1 q_2 \dots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

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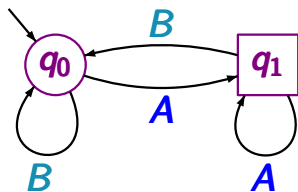
run π is **accepting** if $\exists i \in \mathbb{N}. q_i \in F$

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- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
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accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

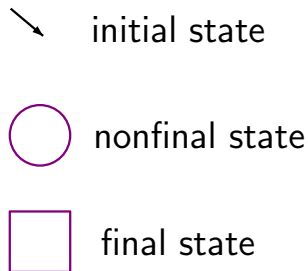
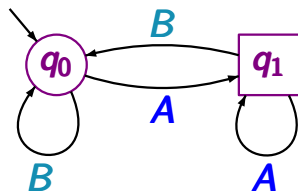
$\mathcal{L}_\omega(\mathcal{A}) \stackrel{\text{def}}{=} \text{set of infinite words over } \Sigma \text{ that have an accepting run in } \mathcal{A}$



↘ initial state

○ nonfinal state

□ final state

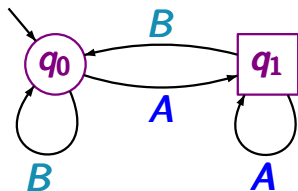


NBA with state space $\{q_0, q_1\}$

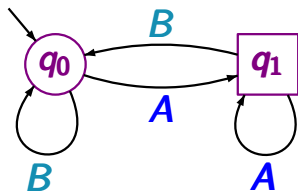
q_0 initial state

q_1 accept state

alphabet $\Sigma = \{A, B\}$

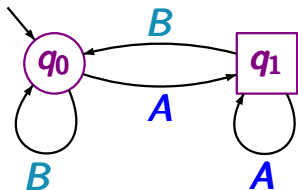


accepted language: ?



accepted language:

set of all infinite words that contain infinitely many **A**'s



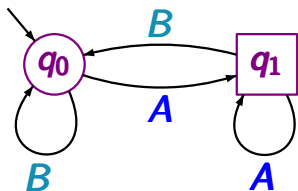
accepted language:

set of all infinite words that
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$$(B^*.A)^\omega$$

Examples for NBA over $\Sigma = \{A, B\}$

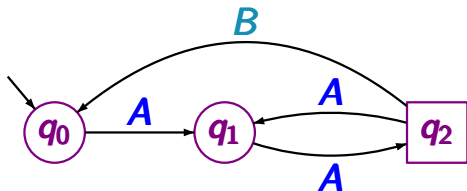
LTLMC3.2-22



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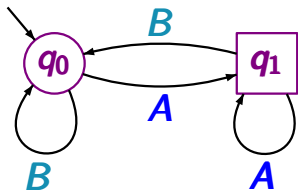
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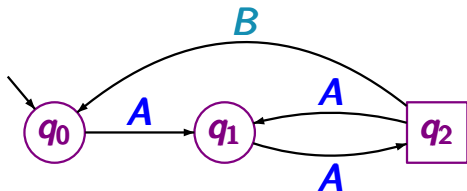
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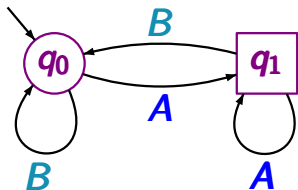
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A**A****B****A****A****B****A****A****B** ...
A**A****A****A****A****A****A****A** ...

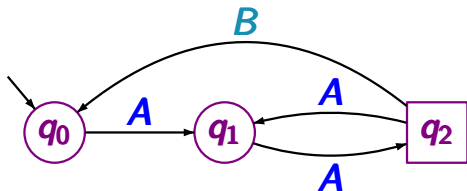
} accepted words



accepted language:

set of all infinite words that contain infinitely many **A**'s

$$(B^*.A)^\omega$$

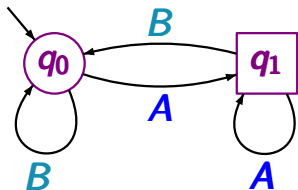


accepted language:

“every **B** is preceded by a positive even number of **A**'s”

$AABAAABAAAB\dots$
 $AAAAAAAAAA\dots$

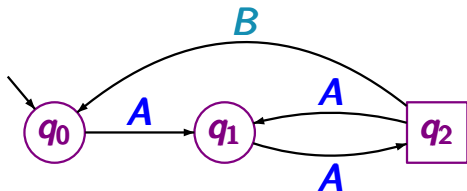
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$$(B^*.A)^\omega$$



accepted language:

“every **B** is preceded by a positive even number of **A**'s”

$$((A.A)^+.B)^\omega + ((A.A)^+.B)^*.A^\omega$$

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of **final states**, also called **accept states**

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

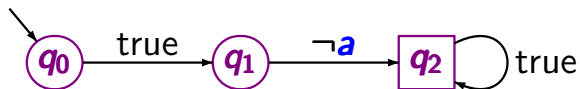
- Q finite set of states
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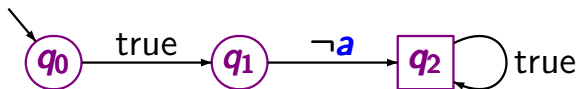
accepted language $\mathcal{L}_w(\mathcal{A})$ is an **LT-property**:

$\mathcal{L}_w(\mathcal{A}) =$ set of infinite words over 2^{AP} that
have an **accepting run** in \mathcal{A}



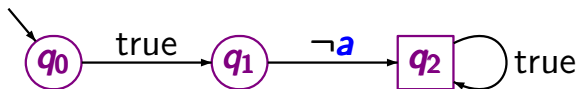
$$\mathcal{L}_\omega(\mathcal{A}) = ?$$

set of atomic propositions $AP = \{a, b\}$

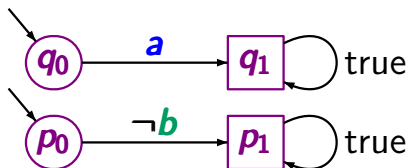


$$\mathcal{L}_\omega(\mathcal{A}) \hat{=} \text{true} \cdot \neg a \cdot \text{true}^\omega$$

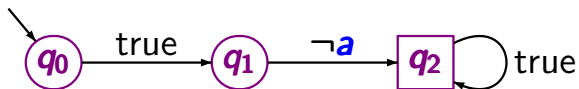
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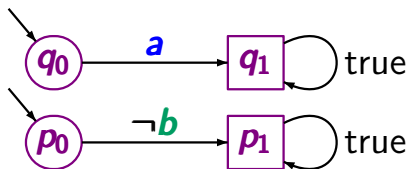
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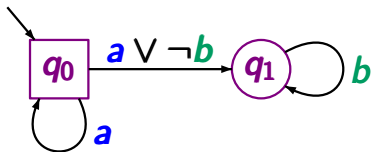
$$\mathcal{L}_\omega(\mathcal{A}) \hat{=} \text{true} \cdot \neg a \cdot \text{true}^\omega$$

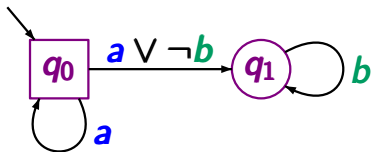


$$(a \vee \neg b) \cdot \text{true}^\omega$$

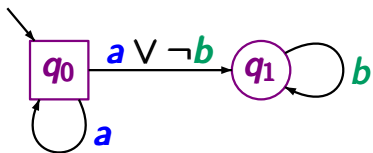
set of atomic propositions $AP = \{a, b\}$

NBA for LT properties

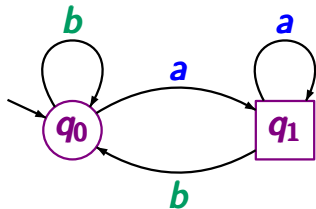


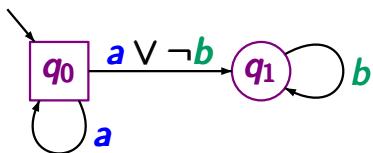


“always a ” $\hat{=} a^\omega$

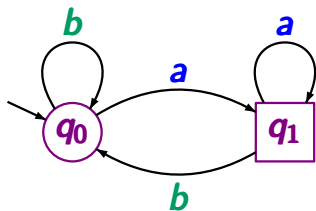


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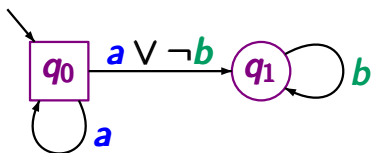




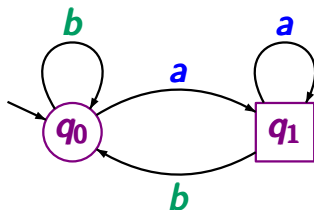
“always a ” $\hat{=} a^\omega$



“infinitely often a and ...”

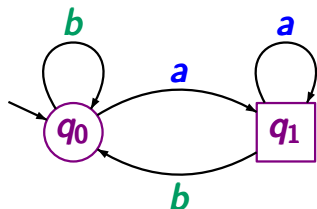


“always a ” $\hat{=} a^\omega$



“infinitely often a and always $a \vee b$ ”

$$\hat{=} ((a \vee b)^* \cdot a)^\omega$$

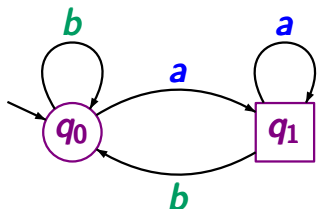


“infinitely often a and
always $a \vee b$ ”

$$((a \vee b)^* \cdot a)^\omega$$

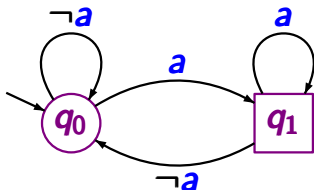
“infinitely often a ”

$$((\neg a)^* \cdot a)^\omega$$



“infinitely often a and
always $a \vee b$ ”

$$((a \vee b)^* \cdot a)^\omega$$



“infinitely often a ”

$$((\neg a)^* \cdot a)^\omega$$

For each NBA \mathcal{A} there is an ω -regular expression γ
with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

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Proof. Let \mathcal{A} be an NBA $(Q, \Sigma, \delta, Q_0, F)$

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Proof. Let \mathcal{A} be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$.
Let $\mathcal{A}_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$.

For each NBA \mathcal{A} there is an ω -regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

Proof. Let \mathcal{A} be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$. Let $\mathcal{A}_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$. Then:

$$\mathcal{L}_\omega(\mathcal{A}) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(\mathcal{A}_{q,p}) (\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\})^\omega$$

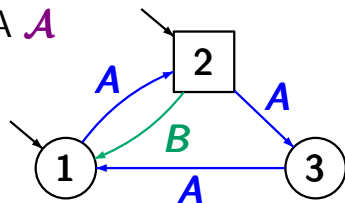
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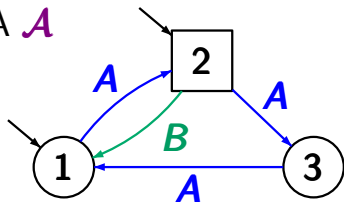
$$\mathcal{L}_\omega(\mathcal{A}) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(\mathcal{A}_{q,p}) (\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\})^\omega$$

is ω -regular as $\mathcal{L}(\mathcal{A}_{q,p})$ and $\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\}$ are regular

NBA \mathcal{A}



NBA \mathcal{A}



$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

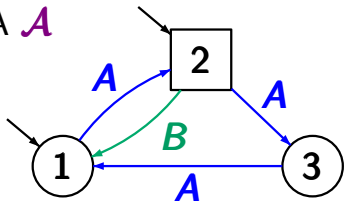
$$L_{12} = \mathcal{L}(A_{12})$$

$$L_{22} = \mathcal{L}(A_{22})$$

$$L'_{22} = L_{22} \setminus \{\varepsilon\}$$

Example: NBA \rightsquigarrow ω -regular expression

LTLMC3.2-26

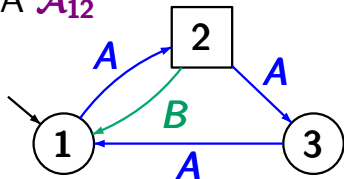
NBA \mathcal{A} 

$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

$$L_{12} = \mathcal{L}(A_{12})$$

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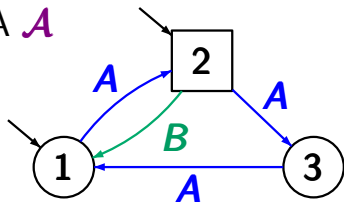
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NFA \mathcal{A}_{12} 

Example: NBA \rightsquigarrow ω -regular expression

LTLMC3.2-26

NBA \mathcal{A}



$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

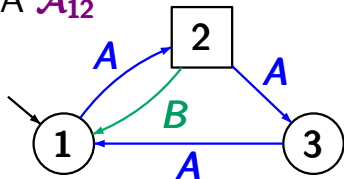
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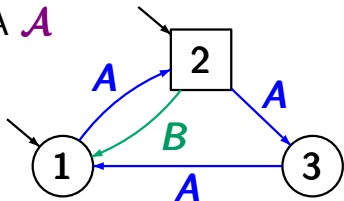
$$L_{12} \hat{=} A.(B.A + A.A.A)^*$$

NFA \mathcal{A}_{12}



Example: NBA \rightsquigarrow ω -regular expression

NBA \mathcal{A}



$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

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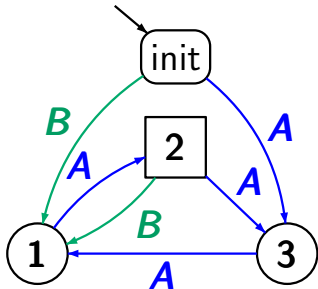
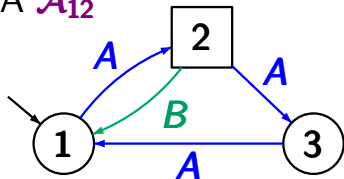
$$L_{22} = \mathcal{L}(\mathcal{A}_{22})$$

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$$L_{12} \hat{=} A.(B.A + A.A.A)^*$$

$$L'_{22} \hat{=} (B.A + A.A.A)^+$$

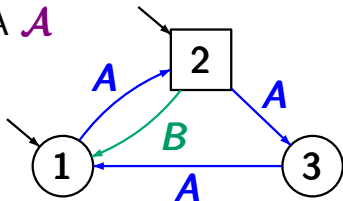
NFA \mathcal{A}_{12}



Example: NBA \rightsquigarrow ω -regular expression

LTLMC3.2-26

NBA \mathcal{A}



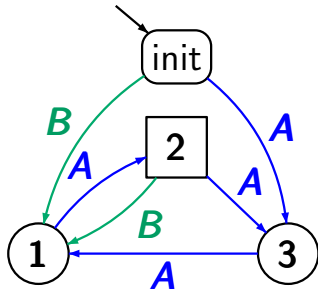
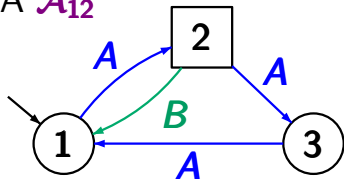
language of \mathcal{A} :

$$A.(B.A + A.A.A)^\omega + (B.A + A.A.A)^\omega$$

$$L_{12} \hat{=} A.(B.A + A.A.A)^*$$

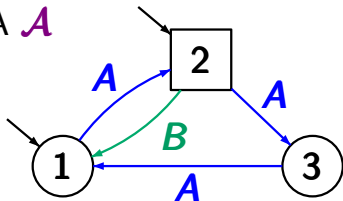
$$L'_{22} \hat{=} (B.A + A.A.A)^+$$

NFA \mathcal{A}_{12}



Example: NBA \rightsquigarrow ω -regular expression

NBA \mathcal{A}



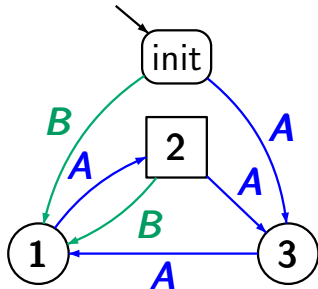
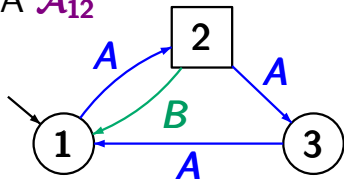
language of \mathcal{A} :

$$\begin{aligned}
 & A.(B.A + A.A.A)^\omega \\
 & + (B.A + A.A.A)^\omega \\
 \equiv & (A + \varepsilon).(B.A + A.A.A)^\omega
 \end{aligned}$$

$$L_{12} \hat{=} A.(B.A + A.A.A)^*$$

$$L'_{22} \hat{=} (B.A + A.A.A)^+$$

NFA \mathcal{A}_{12}



For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an **NBA** \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

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Proof.

For each ω -regular expression

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Proof. consider **NFA** \mathcal{A}_i for α_i and \mathcal{B}_i for β_i

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Proof. consider **NFA** \mathcal{A}_i for α_i and \mathcal{B}_i for β_i

- construct **NBA** \mathcal{B}_i^ω for β_i^ω

For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an **NBA** \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

Proof. consider **NFA** \mathcal{A}_i for α_i and \mathcal{B}_i for β_i

- construct **NBA** \mathcal{B}_i^ω for β_i^ω
- construct **NBA** $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$

For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an **NBA** \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

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- construct **NBA** \mathcal{B}_i^ω for β_i^ω
- construct **NBA** $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$
- construct **NBA** for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an NBA \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

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- construct NBA \mathcal{B}_i^ω for β_i^ω
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$
- construct **NBA** for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$



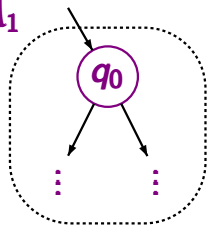
NBA are closed under union

LTLMC3.2-28

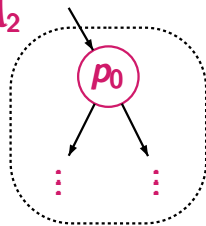
NBA are closed under union

LTLMC3.2-28

NBA \mathcal{A}_1



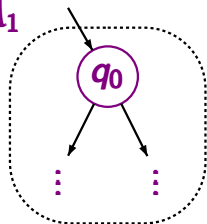
NBA \mathcal{A}_2



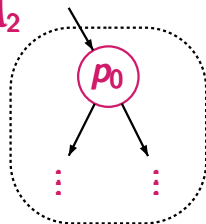
NBA are closed under union

LTLMC3.2-28

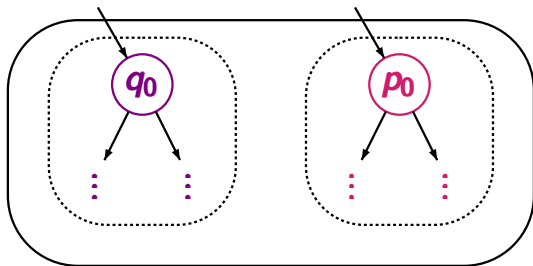
NBA \mathcal{A}_1



NBA \mathcal{A}_2



NBA for $\mathcal{L}_w(\mathcal{A}_1) \cup \mathcal{L}_w(\mathcal{A}_2)$



For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an NBA \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

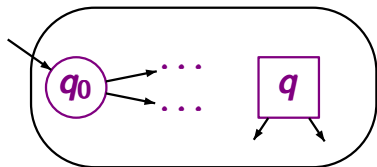
Proof. consider NFA \mathcal{A}_i for α_i and \mathcal{B}_i for β_i

- construct NBA \mathcal{B}_i^ω for β_i^ω
- construct **NBA** $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$ ←
- construct NBA for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

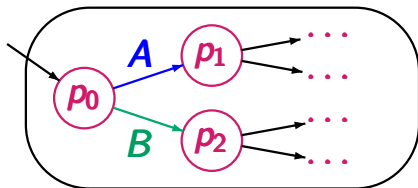
Concatenation of an NFA and an NBA

LTLMC3.2-29

NFA \mathcal{A}_1



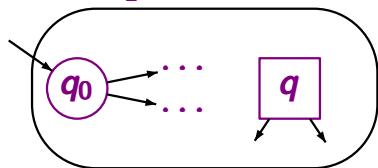
NBA \mathcal{A}_2



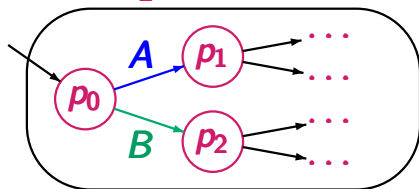
Concatenation of an NFA and an NBA

LTLMC3.2-29

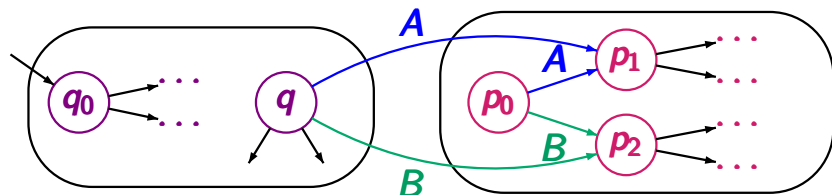
NFA \mathcal{A}_1



NBA \mathcal{A}_2



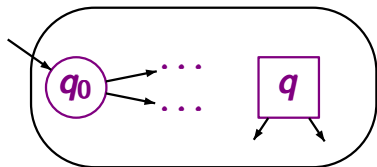
NBA for $\mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}_\omega(\mathcal{A}_2)$:



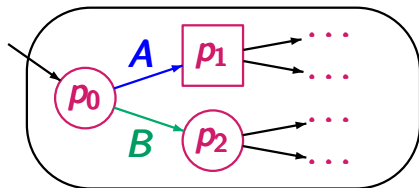
Concatenation of an NFA and an NBA

LTLMC3.2-29

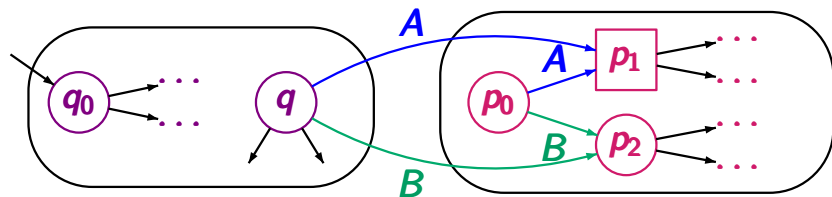
NFA \mathcal{A}_1



NBA \mathcal{A}_2

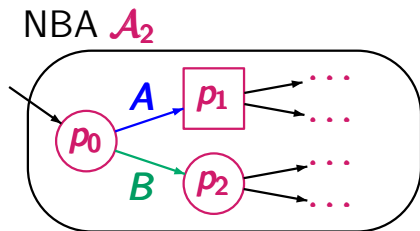
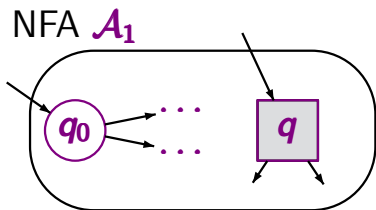


NBA for $\mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}_\omega(\mathcal{A}_2)$:

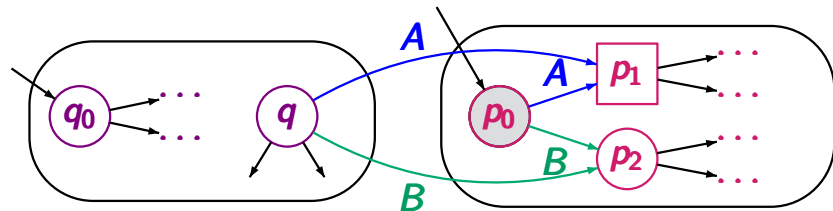


accept states as in \mathcal{A}_2

Concatenation of an NFA and an NBA



NBA for $\mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}_\omega(\mathcal{A}_2)$:



accept states as in \mathcal{A}_2

For each ω -regular expression

$$\gamma = \alpha_1 \cdot \beta_1^\omega + \dots + \alpha_n \cdot \beta_n^\omega$$

there exists an NBA \mathcal{A} with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

Proof. consider NFA \mathcal{A}_i for α_i and \mathcal{B}_i for β_i

- construct NBA \mathcal{B}_i^ω for β_i^ω
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i \cdot \beta_i^\omega$
- construct NBA for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$



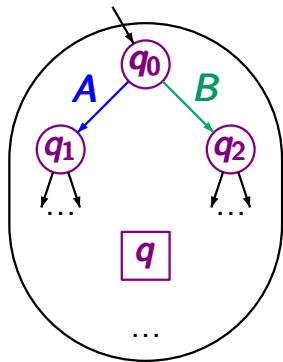
ω -operator for NFA

LTLMC3.2-30

NFA \mathcal{A} for language
 $L \subseteq \Sigma^+$



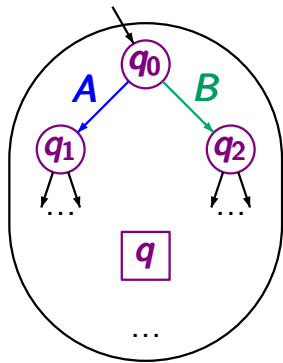
NBA \mathcal{A}^ω for language
 $L^\omega \subseteq \Sigma^\omega$



ω -operator for NFA

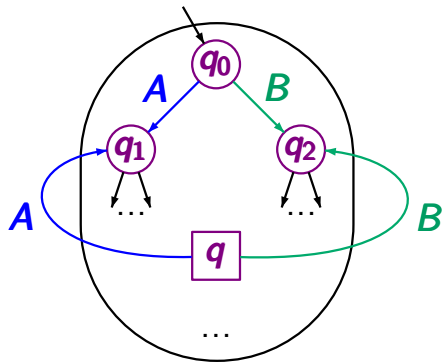
LTLMC3.2-30

NFA \mathcal{A} for language
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\rightsquigarrow

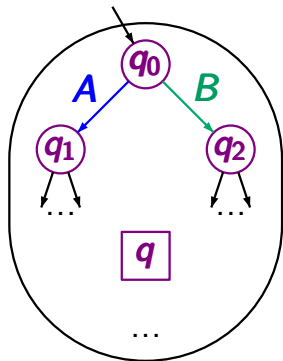
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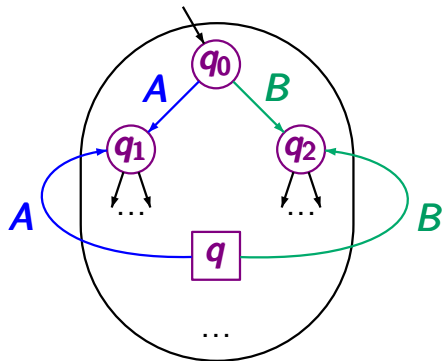
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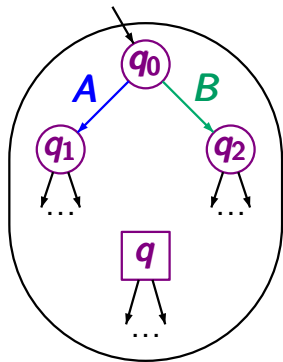


wrong !

ω -operator for NFA

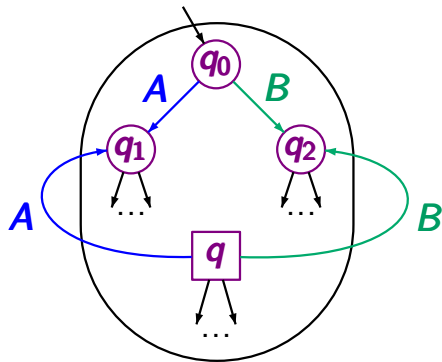
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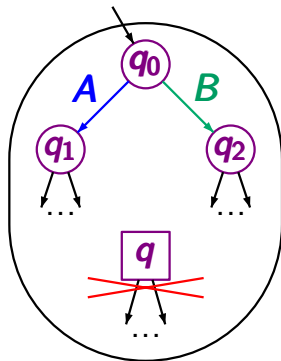


wrong !

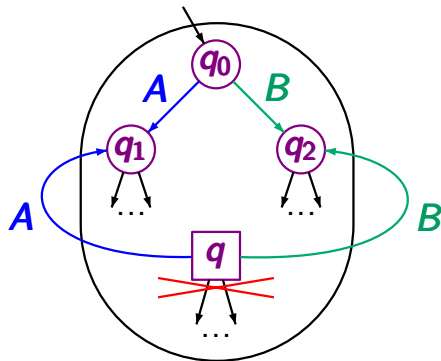
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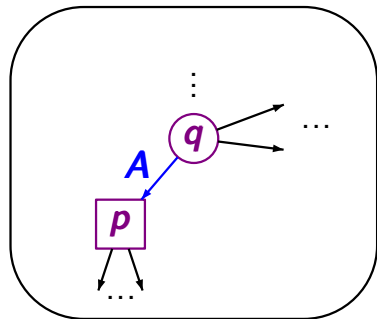
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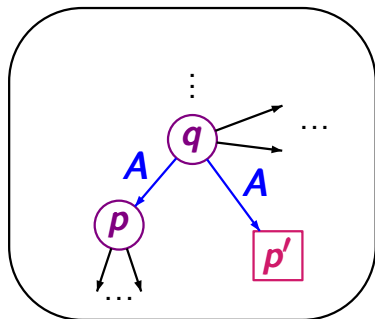
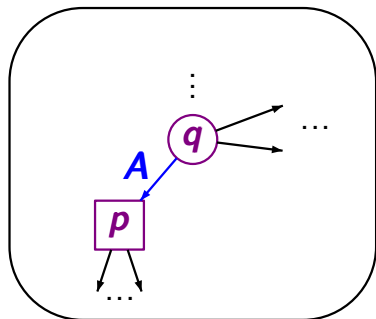
wrong !

... correct, if $\delta(q, x) = \emptyset \quad \forall q \in F \quad \forall x \in \Sigma$

NFA \mathcal{A} for language $L \subseteq \Sigma^+$ \implies NFA \mathcal{B} for L s.t. all final states are terminal



NFA \mathcal{A} for language $L \subseteq \Sigma^+$ \implies NFA \mathcal{B} for L s.t. all final states are terminal



... add a new final state p' ...

ω -operator for NFA

LTLMC3.2-31

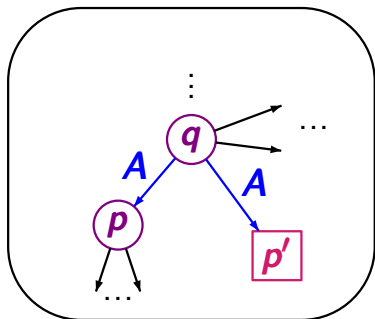
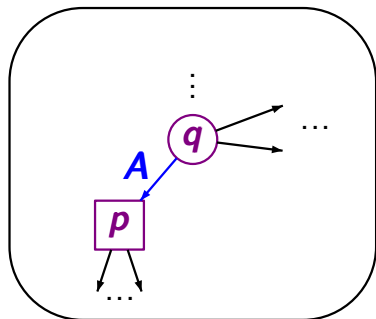
NFA \mathcal{A} for language
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NFA \mathcal{B} for L s.t. all
final states are terminal



NBA \mathcal{B}^ω



... add a new final state p' ...

ω -operator for NFA

LTLMC3.2-31

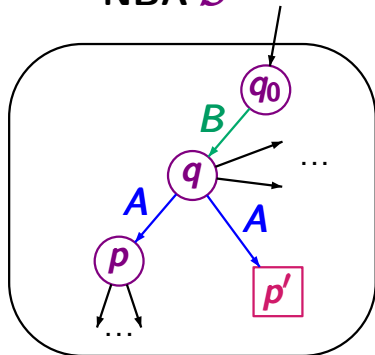
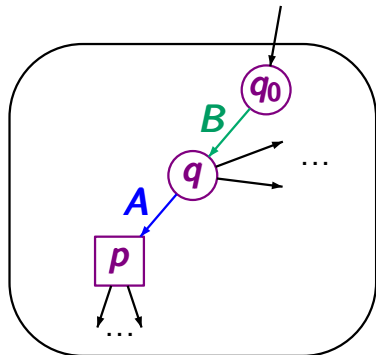
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LTLMC3.2-31

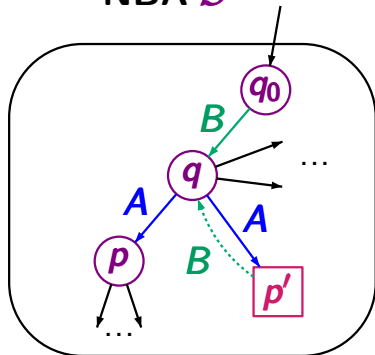
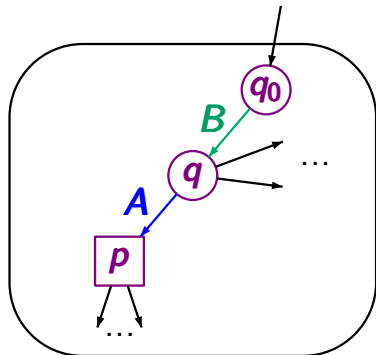
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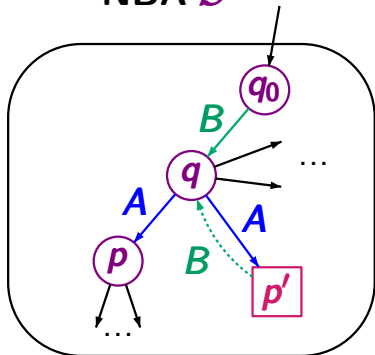
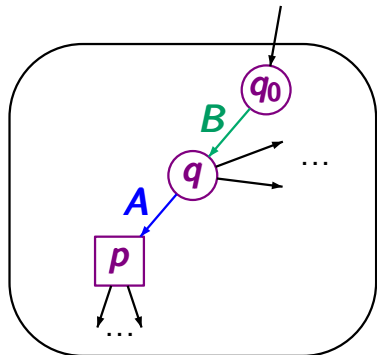
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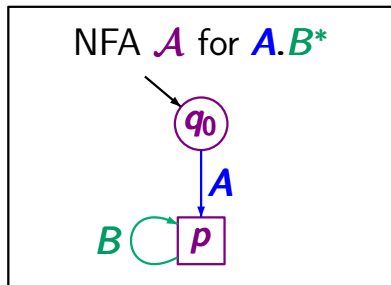
NBA \mathcal{B}^ω



$$\mathcal{L}(\mathcal{A})^\omega = \mathcal{L}_\omega(\mathcal{B}^\omega)$$

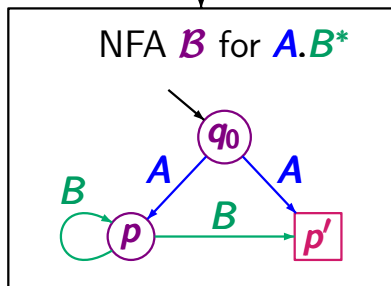
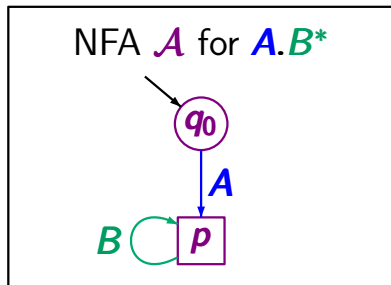
Example: ω -operator for NFA

LTLMC3.2-32



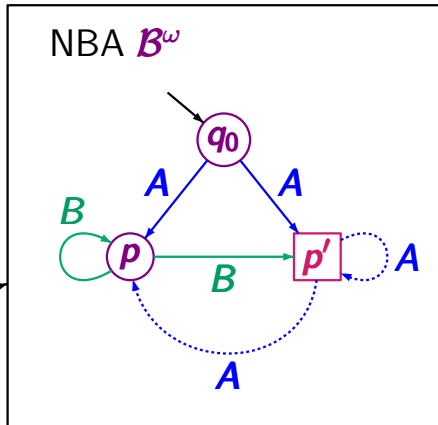
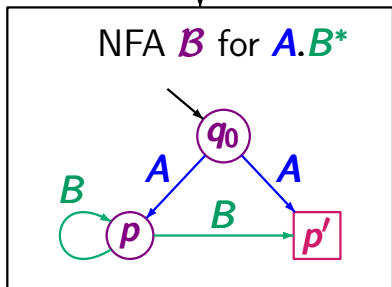
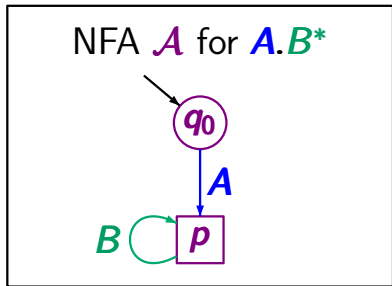
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LTLMC3.2-32



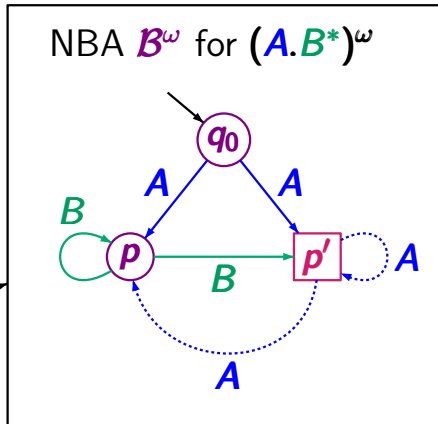
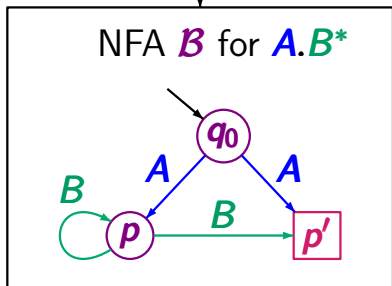
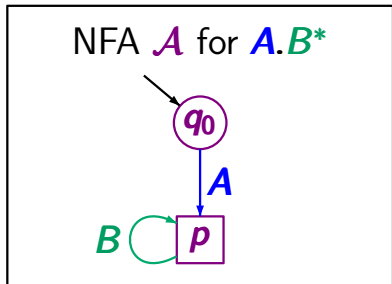
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LTLMC3.2-32



Example: ω -operator for NFA

LTLMC3.2-32



- (1) For each NBA \mathcal{A} there exists an ω -regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$
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Corollary:

If E be an LT property then:

E is ω -regular iff $E = \mathcal{L}_\omega(\mathcal{A})$ for some NBA \mathcal{A}

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Corollary:

If E be an LT property, i.e., $E \subseteq (2^{AP})^\omega$, then:

E is ω -regular iff $E = \mathcal{L}_\omega(\mathcal{A})$ for some NBA \mathcal{A} over the alphabet 2^{AP}

remind: Kleene's theorem for regular languages:

The class of **regular languages** is closed under

- union, intersection, complementation
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The class of **ω -regular languages** is closed under **union, intersection** and **complementation**.

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- *union:*
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- *complementation*:
much more difficult than for NFA,
via other types of ω -automata

Nonemptiness for NBA

LTLMC3.2-NBA-EMPTINESS

given: NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

question: does $\mathcal{L}_\omega(\mathcal{A}) \neq \emptyset$ hold ?

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \emptyset \quad \text{iff} \quad \exists q_0 \in Q_0 \exists p \in F \exists x \in \Sigma^* \exists y \in \Sigma^+.$$
$$p \in \delta(q_0, x) \cap \delta(p, y)$$

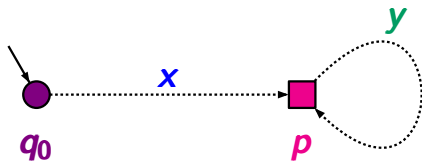
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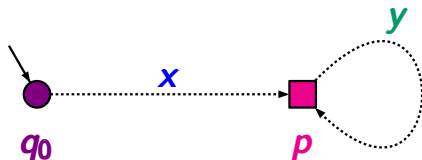
there exists a reachable accept state $p \in F$
that belongs to a cycle



Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\begin{aligned} \mathcal{L}_\omega(\mathcal{A}) \neq \emptyset & \text{ iff } \exists q_0 \in Q_0 \exists p \in F \exists x \in \Sigma^* \exists y \in \Sigma^+. \\ & p \in \delta(q_0, x) \cap \delta(p, y) \\ & \text{ iff } \text{ there exist finite words } x, y \in \Sigma^* \\ & \text{ s.t. } y \neq \varepsilon \text{ and } xy^\omega \in \mathcal{L}_\omega(\mathcal{A}) \end{aligned}$$



Nonemptiness for NBA

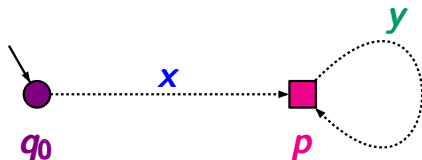
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s.t. $y \neq \varepsilon$ and $xy^\omega \in \mathcal{L}_\omega(\mathcal{A})$

↑
“ultimately periodic words”



Nonemptiness for NBA

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The emptiness problem for NBA is solvable by means of graph algorithms in time $\mathcal{O}(\text{poly}(\mathcal{A}))$

A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- \mathcal{A} has a unique initial state,
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

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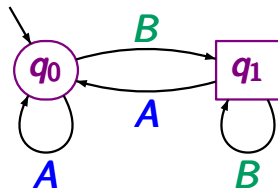
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notation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$

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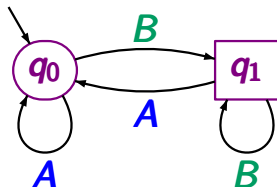


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DBA for “infinitely often B ”

alphabet $\Sigma = \{A, B\}$

well-known:

the powerset construction for the
determinization (and complementation) of
finite automata (NFA)

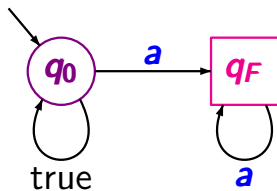
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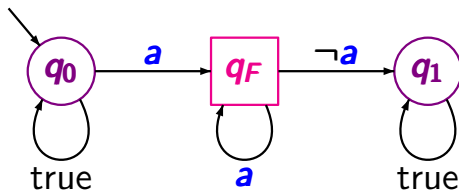
question:

does the powerset construction also work for
Büchi automata (NBA) ?

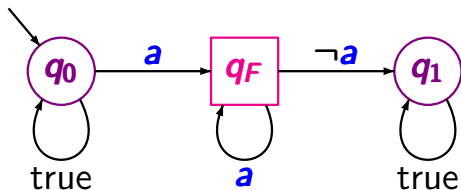
NBA for “eventually forever a ”



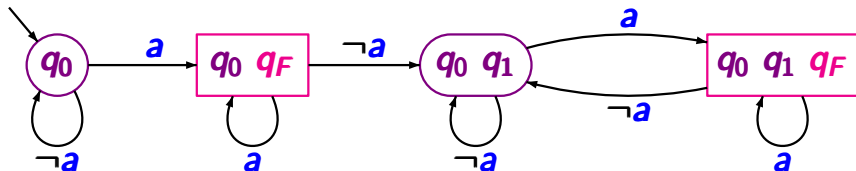
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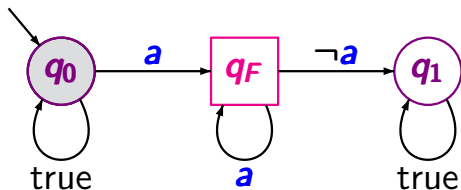
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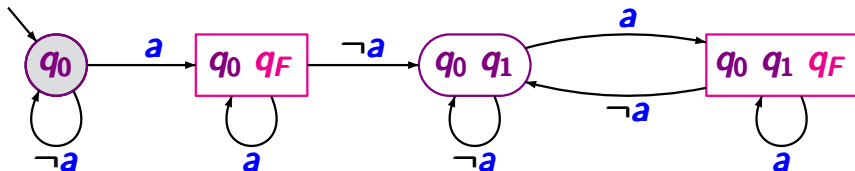
powerset construction



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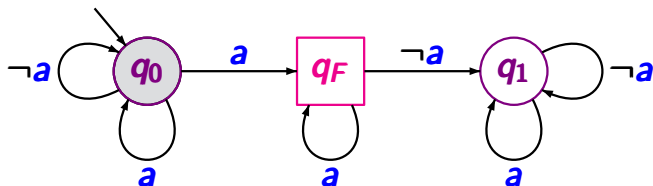


powerset construction

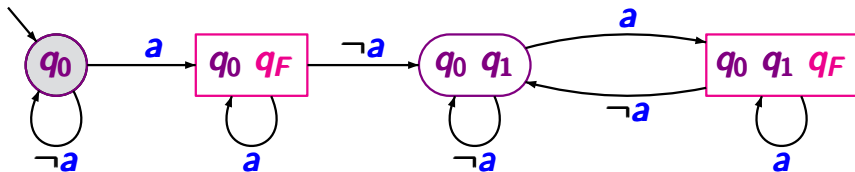


e.g., $\delta(q_0, a) = \{q_0, q_F\}$ and $\delta(q_0, \neg a) = \{q_0\}$

NBA for “eventually forever a ”

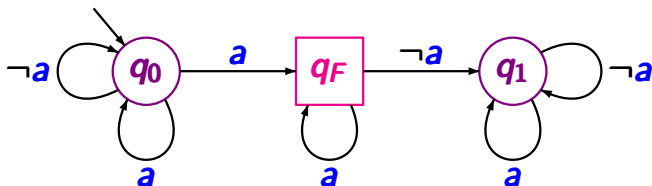


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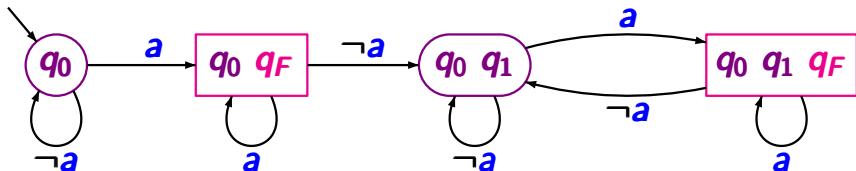


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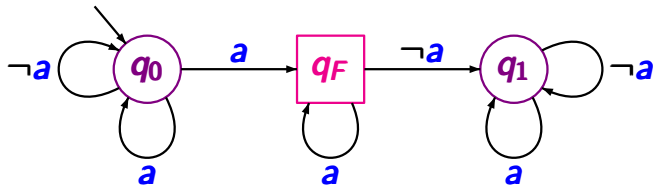
powerset construction



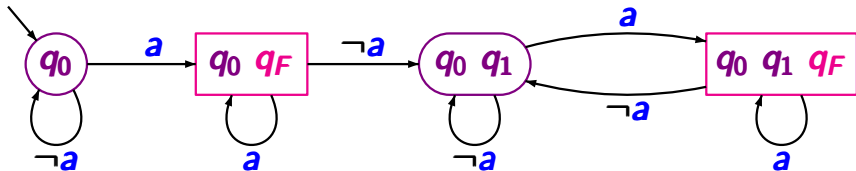
DBA for “infinitely often a ”

Powerset construction ← fails for NBA

NBA for “eventually forever a ”



powerset construction



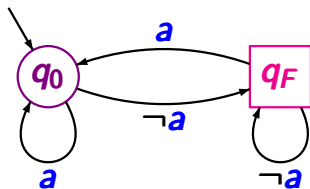
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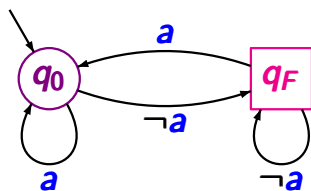
DFA can be complemented by
complementation of the acceptance set

question:

does this also work for **DBA** ?

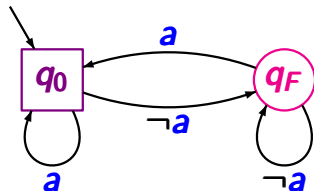


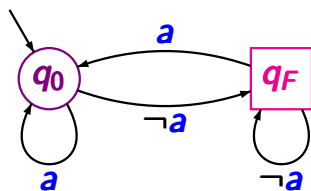
DBA for
“infinitely often $\neg a$ ”



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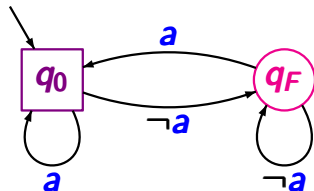
complement automaton



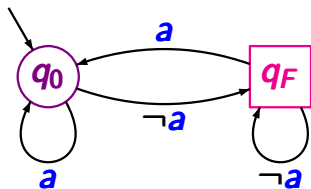


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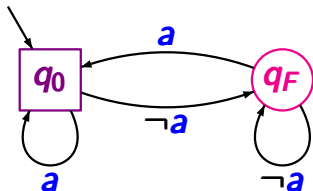


DBA for
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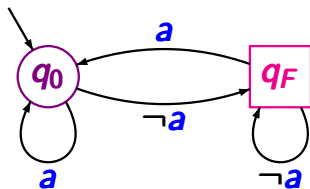


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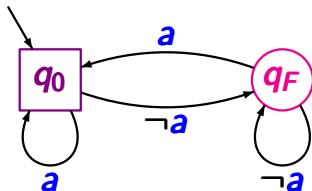


DBA for
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DBA for
“infinitely often a ”

There is **no DBA** for the LT-property
“eventually forever a ”

There is no DBA \mathcal{A} over the alphabet $\Sigma = \{A, B\}$
such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega((A + B)^* \cdot A^\omega)$

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$(A^* \cdot B)^\omega$ “infinitely many B 's” DBA-recognizable

$(A + B)^* \cdot A^\omega$ “only finitely many B 's”
not DBA-recognizable

A generalized nondeterministic Büchi automaton is a tuple

$$\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$$

where Q, Σ, δ, Q_0 are as in NBA, but \mathcal{F} is a set of **accept sets**, i.e., $\mathcal{F} \subseteq 2^Q$.

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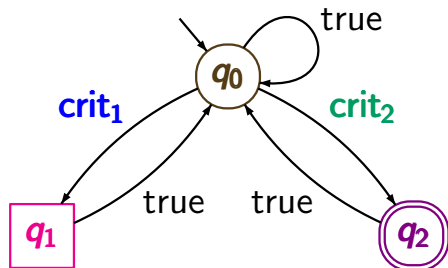
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accepted language:

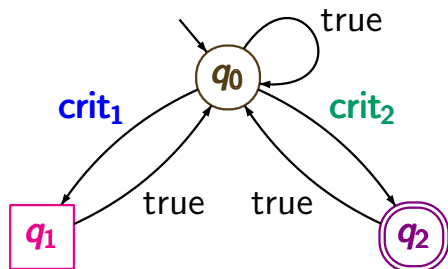
$$\mathcal{L}_\omega(\mathcal{G}) \stackrel{\text{def}}{=} \{ \sigma \in \Sigma^\omega : \sigma \text{ has an accepting run in } \mathcal{G} \}$$

GNBA \mathcal{G} over $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$



$$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$$

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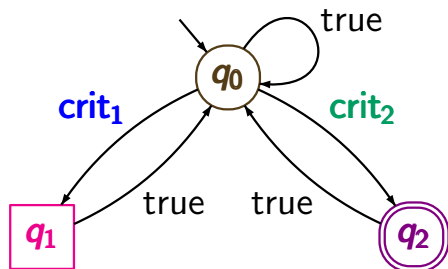


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specifies the LT-property

“infinitely often **crit₁** and infinitely often **crit₂**”

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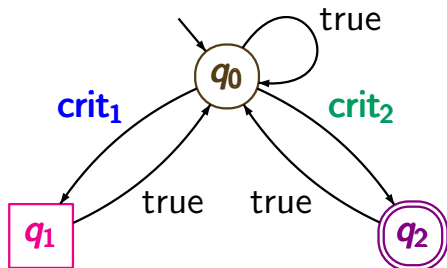


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note: $q_0 \xrightarrow{A} q_1$ implies $A \models \text{crit}_1$

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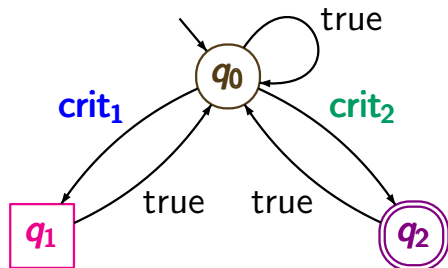
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hence: if $A_0 A_1 A_2 \dots \in \mathcal{L}_\omega(\mathcal{G})$ then

$$\exists^\infty i \geq 0. \text{crit}_1 \in A_i \wedge \exists^\infty i \geq 0. \text{crit}_2 \in A_i$$

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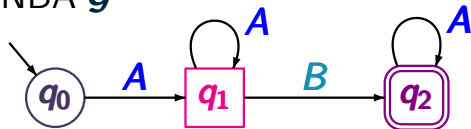


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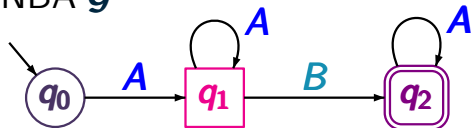
all words $A_0 A_1 A_2 \dots \in \Sigma^\omega$ s.t. $\exists i \geq 0. \text{crit}_1 \in A_i$ and $\exists i \geq 0. \text{crit}_2 \in A_i$ have an accepting run of the form:

$q_0 \dots q_0 q_1 q_0 \dots q_0 q_2 q_0 \dots q_0 q_1 q_0 \dots q_0 q_2 \dots$

GNBA \mathcal{G}

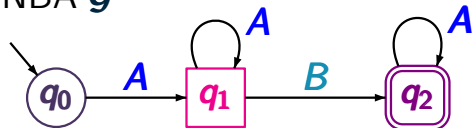


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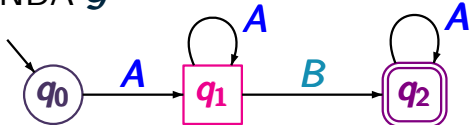
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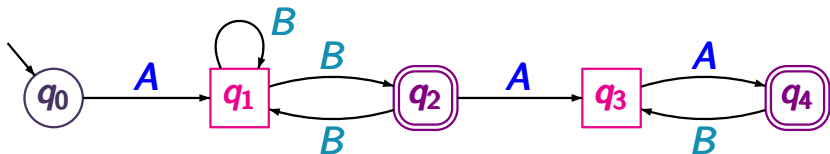
Examples: GNBA over $\Sigma = \{A, B\}$

LTLMC3.2-41

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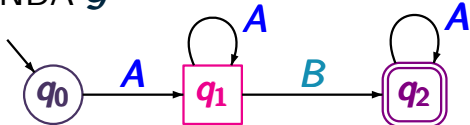
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GNBA \mathcal{G}' with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$ 

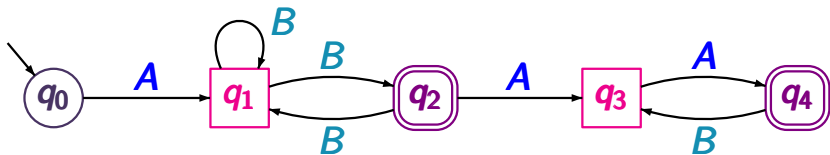
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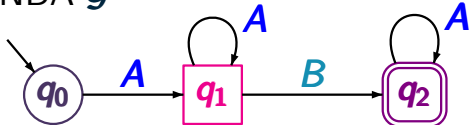
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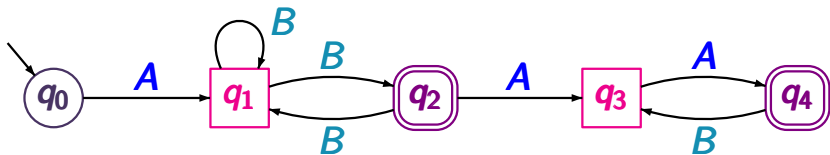
GNBA \mathcal{G}



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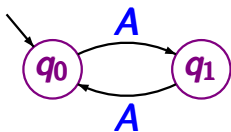


accepted language: $A.B^\omega + A.B^+.A.(A.B)^\omega$

Empty acceptance condition

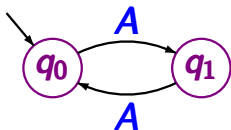
LTLMC3.2-42

NBA \mathcal{A} over $\Sigma = \{A, B\}$:



acceptance set $F = \emptyset$

GNBA \mathcal{G} over $\Sigma = \{A, B\}$:

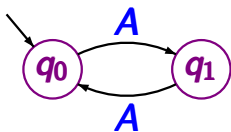


set of acceptance sets
 $\mathcal{F} = \emptyset$

Empty acceptance condition

LTLMC3.2-42

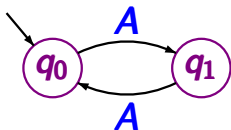
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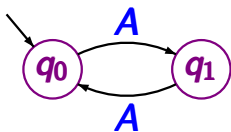
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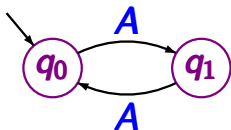
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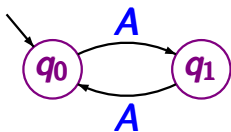
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LTLMC3.2-42

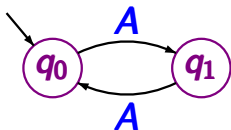
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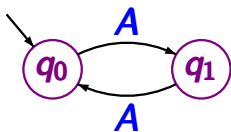
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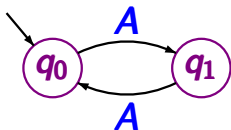
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$$\mathcal{L}_\omega(\mathcal{G}) = \left\{ \begin{array}{l} \text{set of all infinite words} \\ \text{that have an infinite run} \end{array} \right.$$

For every GNBA \mathcal{G} there exists a GNBA \mathcal{G}' such that

- $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{G}')$
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correct

$$\text{GNBA } \mathcal{G} = (Q, \Sigma, \delta, Q_0, \emptyset)$$



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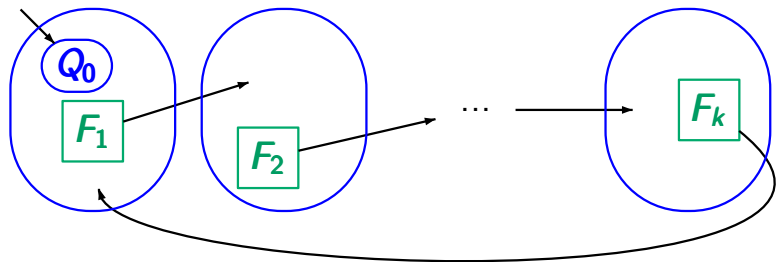
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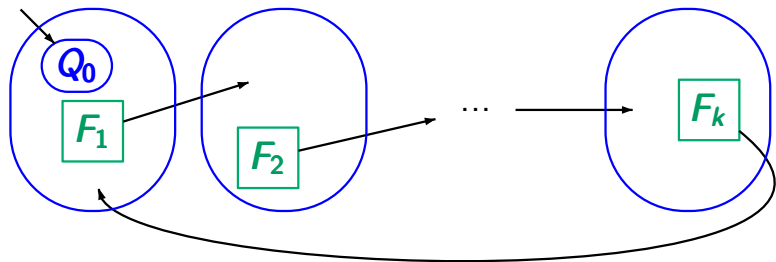
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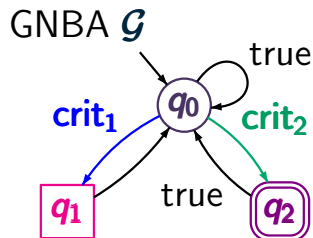
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size of the NBA: $size(\mathcal{A}) = \mathcal{O}(size(\mathcal{G}) \cdot |\mathcal{F}|)$

Example: from GNBA to NBA

LTLMC3.2-45

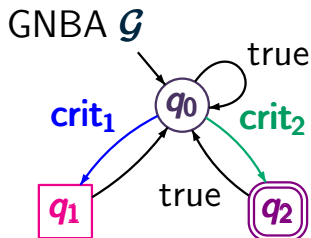


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infinitely often crit_1 and
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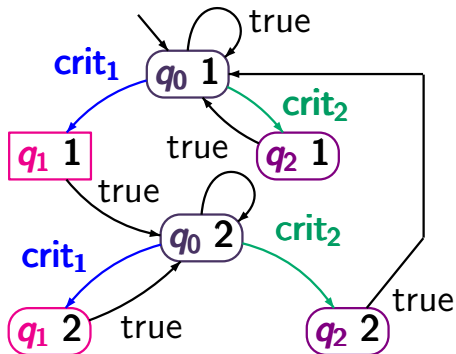
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NBA \mathcal{A}

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via some product construction
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$$\left. \begin{aligned} \mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{aligned} \right\} \text{two NBA}$$

goal: define an NBA \mathcal{A} s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

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recall:

intersection for finite automata NFA \mathcal{A}_1 and \mathcal{A}_2 is realized by a **product construction** that

- runs \mathcal{A}_1 and \mathcal{A}_2 in parallel (synchronously)
- checks whether both end in a final state

$$\left. \begin{array}{l} \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{array} \right\} \text{two NBA}$$

goal: define an NBA \mathcal{A} s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

idea: define $\mathcal{A}_1 \otimes \mathcal{A}_2$ as for NFA, i.e.,

- \mathcal{A}_1 and \mathcal{A}_2 run in parallel (synchronously)
- and check whether both are accepting

$$\left. \begin{array}{l} \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{array} \right\} \text{two NBA}$$

goal: define an NBA \mathcal{A} s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

idea: define $\mathcal{A}_1 \otimes \mathcal{A}_2$ as for NFA, i.e.,

- \mathcal{A}_1 and \mathcal{A}_2 run in parallel (synchronously)
- and check whether both are accepting



i.e., both F_1 and F_2 are visited infinitely often

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\rightsquigarrow product of \mathcal{A}_1 and \mathcal{A}_2 yields a **GNBA**

$$\left. \begin{aligned} \mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{aligned} \right\} \text{two NBA}$$

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$$\text{GNBA } \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

- state space $Q = Q_1 \times Q_2$

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$$\text{GNBA } \mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

- state space $Q = Q_1 \times Q_2$
- alphabet Σ

$$\left. \begin{aligned} \mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{aligned} \right\} \text{two NBA}$$

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- transition relation:

$$\delta(\langle q_1, q_2 \rangle, A) = \{ \langle p_1, p_2 \rangle : p_1 \in \delta_1(q_1, A), p_2 \in \delta_2(q_2, A) \}$$

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GNBA $\mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2 \rightsquigarrow$ equivalent NBA \mathcal{A}

- state space $Q = Q_1 \times Q_2$
- alphabet Σ
- set of initial states: $Q_0 = Q_{0,1} \times Q_{0,2}$
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Summary: ω -regular languages

LTLMC3.2-45C

The class of ω -regular languages agrees with

- the class of languages given by ω -regular expressions
- the class of **NBA**-recognizable languages
- the class of **GNBA**-recognizable languages

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but **DBA** are strictly less expressive

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The class of ω -regular languages is closed under union, intersection and complementation.