Model Checking
Exercises with (Some) Solutions

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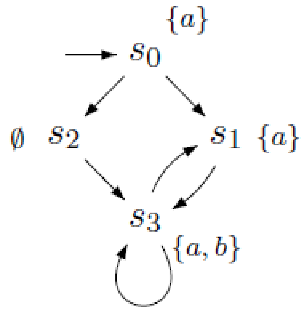
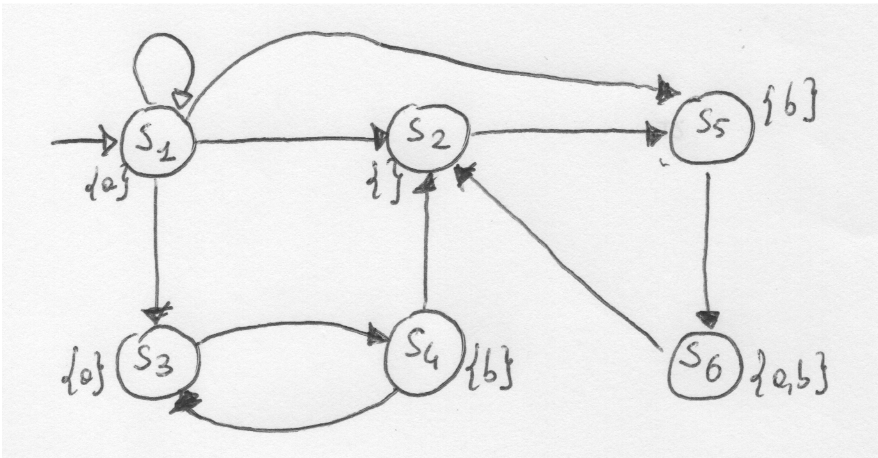
Contents

1 Linear Time Properties

Exercise 1.1. *Give the traces on the set of AP=a,b of the following transition system:*



Exercise 1.2. *Consider the following transition system:*



1. *Define formally the traces on the alphabet* 2^{AP}*, where* $\mathrm{AP} = \{a, b\}$

Exercise 1.3. *Consider the set AP of atomic propositions defined by* $AP = \{x = 0, x > 1\}$ *and consider a non -terminating sequential computer program P that manipulates the variable* x*.*
Formulate the following informally stated properties as LT properties:

a) *false.*
b) *initially* x *is equal to zero.*
c) *initially* x *differs from zero .*
d) *initially* x *is equal to zero, but at some point* x *exceeds one.*
e) x *exceeds one only finitely many times.*
f) x *exceeds one infinitely often.*
g) *the value of* x *alternates between zero and two.*
h) *true*

Exercise 1.4. *Consider the set of atomic propositions* $\mathrm{AP} = \{a, b, c\}$. *Consider the following linear time properties informally stated:*

1. *initially a holds and b does not hold*
2. *c holds only finitely many times*
3. *from some point on the truth value of a alternates between true and false*
4. *whenever c holds, then a holds afterwards*
5. *b holds infinitely many times and whenever b holds then c holds afterwards*
6. *whenever c holds then a or b holds*
7. *a holds only finitely many times and c holds infinitely many times*
8. *whenever a holds then b and c holds after one step*
9. *never a and b hold at the same time and eventually c holds*
10. *at any point the number of times a held in the past is always greater than or equal to the number of times b held in the past.*

For each property above, (a) formally write it as a set of infinite traces on 2^{AP} *and (b) determine whether it is a safety, liveness or mixed (safety and liveness) linear time property. Justify your answers!*
Hint: you may use the special quantifiers $\overset{\infty}{\forall} i$ *("for nearly all i") and* $\overset{\infty}{\exists} i$ *("there exists infinitely many is") as they are defined in the book.*

Exercise 1.5. *Give an algorithm (in pseudo–code) for invariant checking such that in case the invariant is refuted, a minimal counterexample, i.e. a counterexample of minimal length, is provided as error indication.*

Exercise 1.6. *Let P denote the set of traces of the form* $A_0A_1A_2... \in (2^{AP})^\omega$ *such that:*

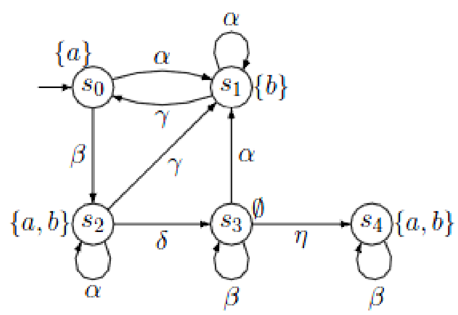
$$\overset{\infty}{\exists} k.\ A_k = \{a, b\} \quad \wedge \quad \exists n \geq 0.\ \forall k > n.\ (a \in A_k \Rightarrow b \in A_{k+1}).$$

Consider the following fairness assumptions with respect to the transition system TS outlined on the right:

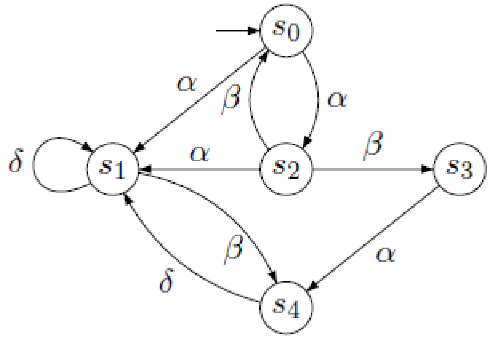
a) $\mathcal{F}_1 = (\{\{\alpha\}\}, \{\{\beta\}, \{\delta, \gamma\}, \{\eta\}\}, \emptyset)$.
Decide whether $TS \models_{\mathcal{F}_1} P$.

b) $\mathcal{F}_2 = \big(\{\{\alpha\}\}, \{\{\beta\}, \{\gamma\}\}, \{\{\eta\}\}\big)$.
Decide whether $TS \models_{\mathcal{F}_2} P$.

Justify your answers!

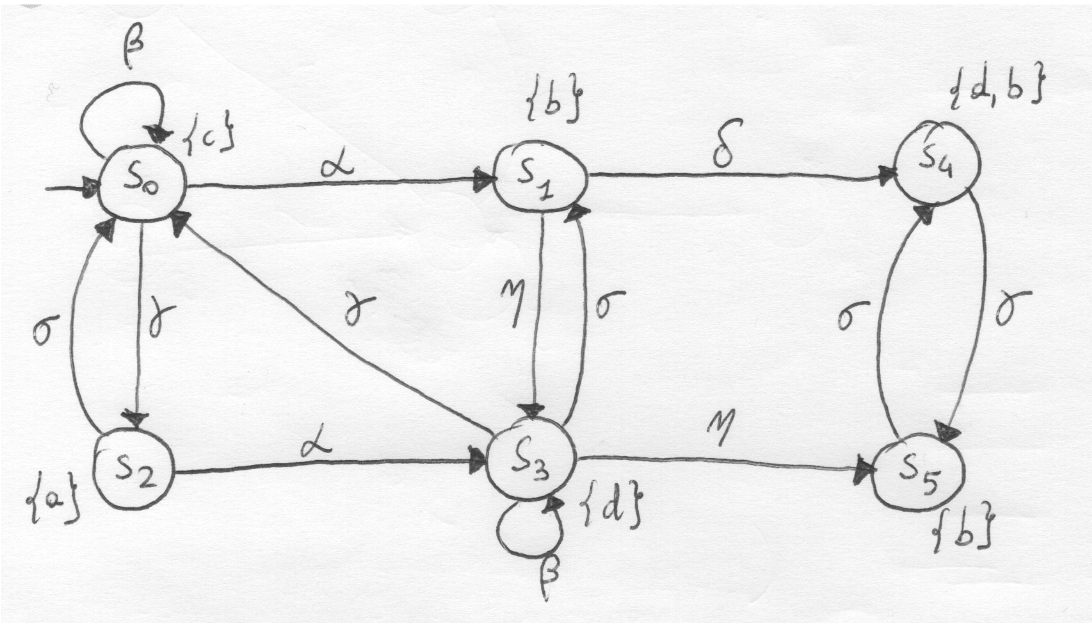


Exercise 1.7. *Consider the transition system TS on the right (where atomic propositions are omitted). Decide which of the following fairness assumption $\mathcal{F}_i$ are realizable for TS.*
justify your answers!



a) $\mathcal{F}_1 = (\{\{\alpha\}\}, \{\{\delta\}\}, \{\{\alpha, \beta\}\})$
b) $\mathcal{F}_2 = (\{\{\delta, \alpha\}\}, \{\{\alpha, \beta\}\}, \{\{\gamma\}\})$
c) $\mathcal{F}_3 = (\{\{\alpha, \delta\}, \{\beta, \}\}, \{\{\alpha, \beta\}\}\{\{\gamma\}\})$

Exercise 1.8. *Consider the following transition system* TS*:*



Consider the following linear time properties, where $\text{AP} = \{a, b, c, d\}$*:*

$$E_1 = \{A_0A_1A_2 \cdots \in (2^{\text{AP}})^\omega \mid \overset{\infty}{\exists}\, i \colon b \in A_i\}$$

$$E_2 = \{A_0A_1A_2 \cdots \in (2^{\text{AP}})^\omega \mid \overset{\infty}{\exists}\, i \colon d \in A_i\}$$

Finally, consider the following fairness assumptions:

$$\begin{aligned} \mathcal{F}_1 &= (\{\{\beta\}\}, \{\{\gamma\}, \{\delta\}\}, \{\{\alpha\}\}) \\ \mathcal{F}_2 &= (\{\{\beta\}\}, \{\{\gamma\}\}, \{\{\alpha\}\}) \\ \mathcal{F}_3 &= (\{\{\beta\}\}, \{\{\gamma\}\}, \{\}) \end{aligned}$$

Decide whether the following model checking statements hold or not:

1. $\text{TS} \models_{\mathcal{F}_1} E_1$
2. $\text{TS} \models_{\mathcal{F}_1} E_2$
3. $\text{TS} \models_{\mathcal{F}_2} E_1$
4. $\text{TS} \models_{\mathcal{F}_2} E_2$
5. $\text{TS} \models_{\mathcal{F}_3} E_1$
6. $\text{TS} \models_{\mathcal{F}_3} E_2$

Justify your answers!

Exercise 1.9. *Let $n \geq 1$. Consider the language $L_n \subseteq \sum^*$ over the alphabet $\sum = \{A, B\}$ that consists of all finite words where the symbol B is on position n from the right, i.e., L_n contains exactly the words $A_1 A_2 ... A_k \in \{A, B\}^*$ where $k \geq n$ and $A_{k-n+1} = B$. For instance, the word $ABBAABAB$ is in L_3.*
a) Construct an NFA An with at most $n + 1$ states such that $L(A_n) = Ln$.
b) Determinize this NFA An using the powerset construction algorithm.

Solutions

Solution of Exercise 1.1

$Traces(TS) = (\{a\}\{a\} + \{a\}\emptyset)(\{a,b\} + \{a,b\}\{a\})^\omega$

How many traces? 2, both infinite.

Solution of Exercise 1.2

All possible paths are of five kinds:

1. s_1^ω
2. $s_1^+(s_5s_6s_2)^\omega$
3. $s_1^+(s_3s_4)^+s_2(s_5s_6s_2)^\omega$
4. $s_1^+(s_3s_4)^\omega$
5. $s_1^+(s_2s_5s_6)^\omega$

The corresponding traces are:

$\{\{a\}^\omega\} \cup \{\{a\}^+(\{b\}\{a,b\}\{\})^\omega\} \cup \{\{a\}^+(\{a\}\{b\})^+(\{\}\{b\}\{a,b\})^\omega\} \cup$

$\{\{a\}^+(\{a\}\{b\})^\omega\} \cup \{\{a\}^+(\{\}\{b\}\{a,b\})^\omega\}$

Solution of Exercise 1.3

$(a)\ P = \varnothing$
$(b)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | x_0 \in A_0\}$
$(c)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | x_0 \notin A_0\}$
$(d)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | x_0 \in A_0 \wedge \exists i : (x > i) \in A_i \wedge i > 0\}$
$(e)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | \exists i \geq 0 : \forall j \geq i,\ (x > i) \notin A_j\}$
$(f)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | \forall i \geq 0 : \exists j \geq i,\ (x > i) \in A_j\}$
$(g)\ P = \{A_0, A_1, A_2... \in (2^{AP})^\omega | (\forall\ (x = 0) \in A_i\ \wedge\ (x > 1) \in A_{i+1}\ \wedge\ i\ mod_2 = 0)\ \vee$
$(\forall\ (x = 0) \in A_i\ \wedge\ (x > 1) \in A_{i+1}\ \wedge\ i\ mod_2 = 1)\}$
$(h)\ P = (2^{AP})^\omega$

Solution of Exercise 1.4

1. The property can be formally stated as
   $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid a \in A_0 \wedge b \notin A_0\}$
   This property is a SAFETY PROPERTY as a bad prefix can be any prefix of a word in $(2^{AP})^\omega$ starting with { } or $\{b\}$ or $\{c\}$ or $\{a,b\}$or $\{b,c\}$or $\{a,b,c\}$.

2. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \overset{\infty}{\forall}\ i \in N,\ c \notin A_i\}$
   This is a LIVENESS PROPERTY because no prefix can be classified as bad because the information on the occurrences of "c" in the tail of the word is missing.

3. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \exists i \in N : \forall j \geq i \; a \in A_j \Leftrightarrow a \notin A_j + i\}$
LIVENESS: no prefix can be classified as bad without the information on the tail of the word.

4. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \forall i \in N : (c \in A_i \implies \exists j \geq i \; : \; a \in A_j)\}$
LIVENESS: as above.

5. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid (\overset{\infty}{\exists} \, i \in N : b \in A_i) \wedge (\forall i \in N : (b \in A_i \implies \exists j \geq i : c \in Aj))\}$
LIVENESS.

6. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \forall i \in N(c \in A_i \implies (a \in A_i \vee b \in A_i))\}$
SAFETY: a bad prefix is, for instance, $\{c\}\{\}\{\}\{\}...$

7. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid (\overset{\infty}{\forall} \, i \in N : a \notin A_i) \wedge (\overset{\infty}{\exists} \, i \in N : c \in A_i)\}$
LIVENESS

8. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \forall i \in N : a \in A_i \implies (b \in A_{i+1} \wedge c \in A_{i+1})\}$
SAFETY: a bad prefix is for instance $\{a\}\{a\}\{a\}...$

9. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid (\forall i \in N : a \in A_i \Leftrightarrow b \notin A_i) \wedge \exists i \in N : c \in A_i\}$ MIXED: a bad prefix for the first part is $\{a, b\}\{\}\{\}...$
The part on " eventually " c cannot have a bad prefix, so it is liveness property.

10. $P = \{A_0, A_1, ... \in (2^{AP})^\omega \mid \forall i \in N : \mid \{0 \leq j \leq i : a \in A_j\} \mid \geq \mid \{0 \leq j \leq i : b \in A_j\} \mid\}$
Where $\mid \{...\} \mid$ is set cardinality.
SAFETY: a bad prefix for example $\{b\}\{\}\{\}...$

Solution of Exercise 1.5

The algorithm works as follows:

Algorithm 1 Invariant Checking using Breadth–First Search

Require: finite transition system TS and propositional formula Φ
Ensure: **true** or the shortest counterexample

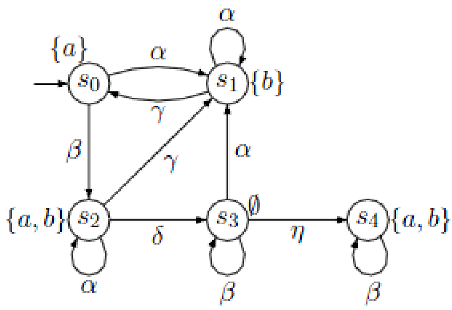
queue of states $Q = \varepsilon$;
finite trace $\hat{\sigma} = \varepsilon$;
set of states R;
set of tuples $P \subseteq S \times S$;

procedure bfs(state s)
enqueue(Q, s);
$P := \{(s, \perp)\}$;
$R := \{s\}$;
while $(Q \neq \varepsilon) \wedge (\text{first}(Q) \models \Phi)$ **do**
  let $p := \text{dequeue}(Q)$;
  for all $p' \in \text{Post}(p) \setminus R$ **do**
    enqueue(Q, p');
    $R := R \cup \{p'\}$;
    $P := P \cup \{(p', p)\}$;
  end for
end while
if $Q \neq \varepsilon$ **then**
  let $p := \text{first}(Q)$;
  while $p \neq \perp$ **do**
    $\hat{\sigma} := p.\hat{\sigma}$;
    let $(p, p') \in P$;
    $p := p'$;
  end while
  return false; shortest counterexample $\hat{\sigma}$;
else
  return true;
end if

Solution of Exercise 1.6

We consider each of the fairness assumptions $\mathcal{F}_i$ for $i \in \{1, 2\}$:

We have $TS \models_{\mathcal{F}_i} P$ iff $FairTraces_{\mathcal{F}_i}(TS) \subseteq P$. Because of $\overset{\infty}{\exists} k.\ A_k = \{a, b\}$, each trace has to visit at least one of s_2 or s_4 infinitely many times.
Additionally, from some point onwards, each a-state must be followed by a state that is annotated with (at least) b.



a) $TS \models_{\mathcal{F}_1} P_2$:

- Any trace that reaches s_4 is not $\mathcal{F}_1$-fair as α is executed only finitely many times. This is in contradiction to our $\mathcal{F}_{1,ucond} = \{\{\alpha\}\}$.
- Therefore $s_3 \xrightarrow{\eta} s_4$ is never taken.
- Because of $\{\eta\} \in \mathcal{F}_{1,strong}$ and because η actions cannot be executed infinitely often (in fact, only once from s_3 to s_4), the state s_3 must not be visited infinitely often.
- We cannot stay in states s_1 or s_2 by only taking transitions $s_1 \xrightarrow{\alpha} s_1$ and $s_2 \xrightarrow{\alpha} s_2$ because of the enabled γ transitions to s_0 or s_1, respectively.
- As β is enabled in s_0, all $\mathcal{F}_1$–fair paths visit exactly s_0, s_1 and s_2 infinitely often.

Therefore $FairTraces_{\mathcal{F}_1}(TS) \subseteq P$ and $TS \models_{\mathcal{F}_1} P$.

b) $TS \not\models_{\mathcal{F}_2} P$:
Consider the path $\pi = (s_0 s_2 s_3 s_1)^\omega$ with its corresponding trace $\sigma = (\{a\}\{a,b\}\emptyset\{b\})^\omega$.
We have $\pi \in FairPaths_{\mathcal{F}_2}(TS)$, but $\sigma \notin P$.
$\Longrightarrow FairTraces_{\mathcal{F}_2}(TS) \not\subseteq P$.

Solution of Exercise 1.7

Realizable fairness assumptions:

a) $\mathcal{F}_1 = (\{\{\alpha\}\}, \{\{\delta\}\}, \{\{\alpha, \beta\}\})$ is not realizable fair. Consider the states s_1 and s_4. There are no $\mathcal{F}_1$ fair path fragments starting from s_1 or s_4, as on each such path fragment, α transitions never occur. This violates the unconditional fairness constraint $\{\{\alpha\}\}$.

b) $\mathcal{F}_2 = (\{\{\delta, \alpha\}\}, \{\{\alpha, \beta\}\}, \{\{\gamma\}\})$ is realizable fair, as the SCC $\{s_1, s_4\}$ is reachable from every state and $(s_1, s_4)^\omega$ is a $\mathcal{F}_2$ fair path fragment.

c) $\mathcal{F}_3 = (\{\{\alpha, \delta\}, \{\beta, \}\}, \{\{\alpha, \beta\}\}\{\{\gamma\}\})$ is realizable fair. Consider the same SCC $\{s_1, s_4\}$ and again the path fragment $(s_1, s_4)^\omega$.

Solution of Exercise 1.8

let's consider F_1.
The unconditional fairness on $\{\beta\}$ excludes the paths in which states S_4 and S_5 are reached.
The strong fairness on $\{\gamma\}$ excludes the paths ending with S_0^ω or S_3^ω.
The strong fairness on $\{\delta\}$ excludes the paths in which s_1 is visited infinitely many times because otherwise the state s_4 is reached.

The weak fairness on $\{\alpha\}$ excludes the paths cycling between states s_0 and s_2.
Thus the only fair paths are those the visit infinitely often the states s_0,s_2 and s_3, but not s_1.
in the light of the observations above we can conclude that $TS \not\models_{F_1} E_1$ and $TS \models_{F_1} E_2$.

let's consider F_2.
The missing strong fairness on $\{\delta\}$ allows also the paths in which state s_1 is visited infinitely often.
However, the paths in which only the states s_0,s_2 and s_3 are still fair, so we have to conclude, as far F_1: $TS \not\models_{F_2} E_1$ and $TS \models_{F_2} E_2$.

let's consider F_3.
The missing weak fairness on $\{\alpha\}$ allows, in addition to the ones fair for F_2, the paths that visit infinitely often only the statess_0 and s_1.
Thus the only fair paths are those the visit infinitely often the states s_0,s_2 and s_3, but not s_1.
Thus, we conclude that $TS \not\models_{F_3} E_1$ and $TS \not\models_{F_3} E_2$.

Solution of Exercise 1.9

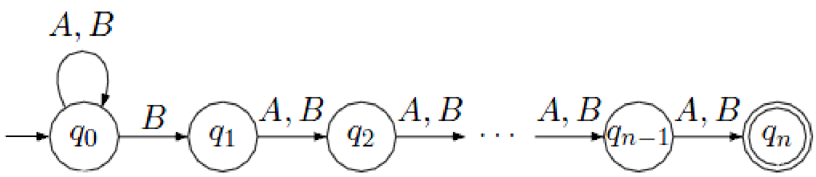
a) Formally, we define the NFA $A_n = (Q_n, \sum, \delta_n, Q_0, F)$ where

- $Q_n = \{q_0, q_1, ..., q_n\}$
- transition relation defined by δ_n:

$$\delta_n(q_0, A) = \{q_0\} \qquad \delta_n(q_0, B) = \{q_0, q_1\}$$
$$\delta_n(q_0, A) = \{q_{q_i+1}\} for 0 < i < n \qquad \delta_n(q_i, B) = \{q_{i+1}\} for 0 < i < n$$

- the set of initial states: $Q_0 = \{q_0\}$
- $F = \{q_n\}$

This can also be outlined as follows:



b) Applying the powerset construction to the NFA A_n yields the DFA $A'_n = (2^{Q_n}, \sum, \delta'_n, \{q_0\}, F'_n)$ where

- the transition function δ'_n is defined (for $k \in \{0, ..., n\}$) as follows:

$$\delta'_n(\{q_0, q_{i1}, ..., q_{ik}\}, A) = \{q_{ij+1} | i_j < n, j \in \{1, ..., k\}\} \cup \{q_0\}$$

$$\delta'_n(\{q_0, q_{i1}, ..., q_{ik}\}, A) = \{q_{ij+1} | i_j < n, j \in \{0, ..., k\}\} \cup \{q_0\}$$

- The acceptance set is given by $F'_n = \{Q' \in 2^{Q_n} | q_n \in Q'\}$