CTL: Syntax, Semantics, Equivalences and Normal Forms

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Topics

- Syntax and Semantics of CTL. Examples.
- Equivalence of CTL formulas. Duality and expansion laws.
- Normal Forms of CTL formulas.

Material

Reading:

Chapter 6 of the book: Sections 6.1 and 6.2

More:

The slides in the following pages are taken from the material of the course "Introduction to Model Checking" held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

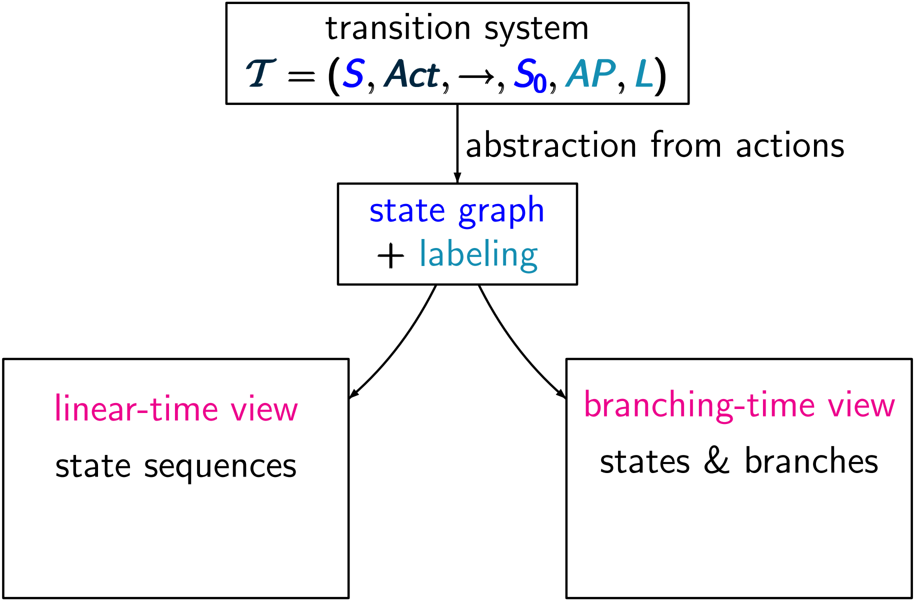
Linear Temporal Logic (LTL)

Computation Tree Logic

Equivalences and Abstraction

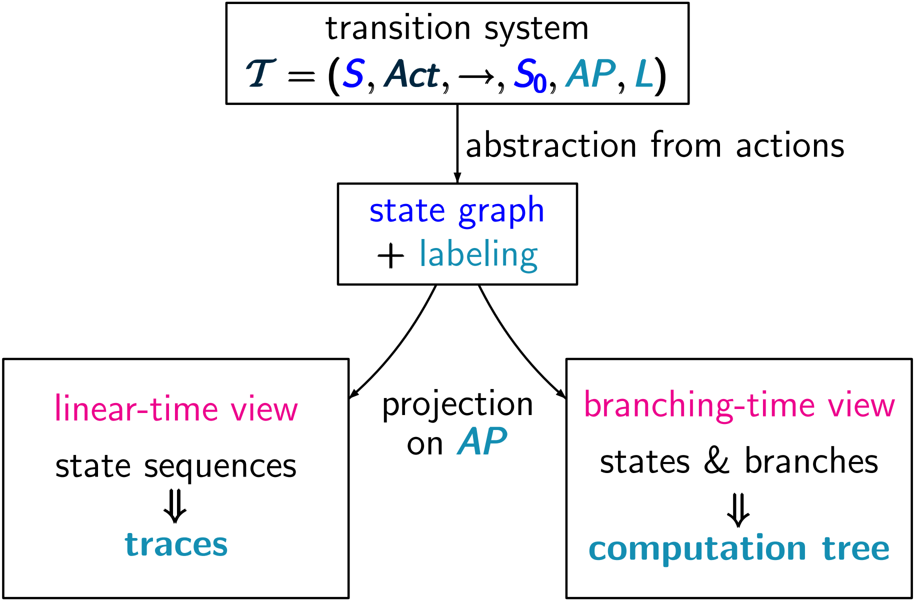
Linear vs branching time

CTLSS4.1-1



transition system
$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
abstraction from actions
state graph
+ labeling
linear-time view
state sequences
branching-time view
states & branches

Linear vs branching time



Computation tree

CTLSS4.1-1B

Computation tree

The computation tree of a transition system $\mathcal{T} = (S, Act, \rightarrow, s_0, AP, L)$ arises by:

- unfolding into a tree
- abstraction from the actions
- projection of the states s to their labels $L(s) \subseteq AP$

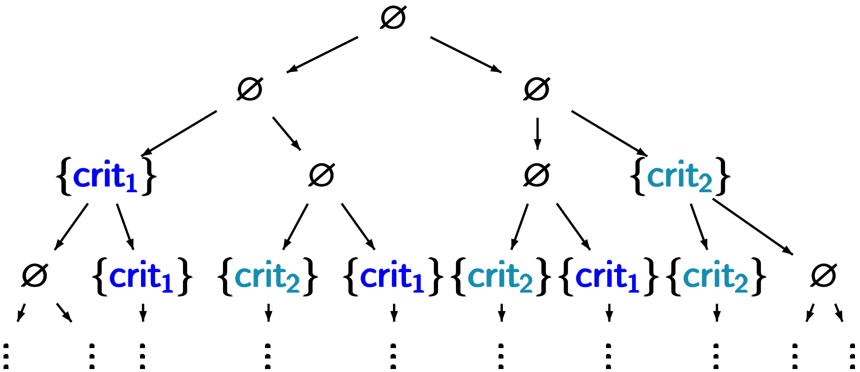
Computation tree

The computation tree of state s_0 in a transition system $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ arises by:

- unfolding $\mathcal{T}_{s_0} = (S, Act, \rightarrow, s_0, AP, L)$ into a tree
- abstraction from the actions
- projection of the states s to their labels $L(s) \subseteq AP$

Example: computation tree

mutual exclusion with semaphore and $AP = \{crit_1, crit_2\}$:



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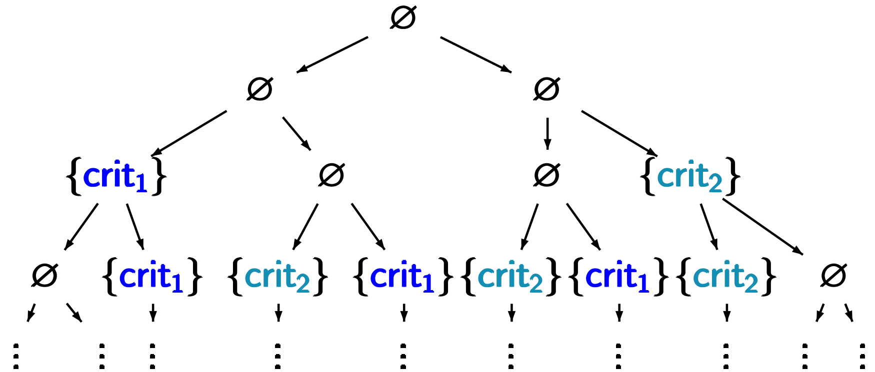

path $\langle nc_1, nc_2 \rangle$ $\langle wait_1, nc_2 \rangle$ $\langle crit_1, nc_2 \rangle$ $\langle crit_1, wait_2 \rangle$ ...

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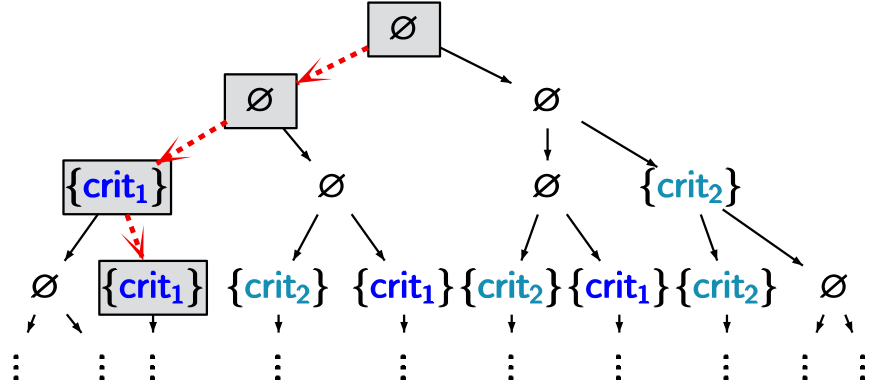

path	$\langle nc_1, nc_2 \rangle$	$\langle wait_1, nc_2 \rangle$	$\langle crit_1, nc_2 \rangle$	$\langle crit_1, wait_2 \rangle$	...
↓	↓	↓	↓	↓	
trace	$\varnothing$	$\varnothing$	$\{crit_1\}$	$\{crit_1\}$	...

Linear vs. branching time

	linear time	branching time
behavior	path based	state based

Linear vs. branching time

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fairness	can be encoded	requires special treatment

Introduction

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Regular Properties

Linear Temporal Logic (LTL)

Computation Tree Logic

- syntax and semantics of CTL ⟵
- expressiveness of CTL and LTL
- CTL model checking
- fairness, counterexamples/witnesses
- CTL^+ and CTL^*

Equivalences and Abstraction

Computation Tree Logic (CTL)

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CTL (state) formulas:

$$\Phi \ ::= \ \textit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

CTL path formulas:

$$\varphi \ ::= \ \bigcirc\Phi \mid \Phi_1 \, \mathsf{U} \, \Phi_2$$

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$$\exists\Diamond\Phi \;\stackrel{\text{def}}{=}\; \exists(\textit{true}\,\mathsf{U}\,\Phi)$$

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always:

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Syntax of CTL

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unconditional process fairness $\forall\Box\forall\Diamond crit_1 \wedge \forall\Box\forall\Diamond crit_2$

Example: 15-puzzle

CTLSS4.1-5

6	8	2	12
4	1	13	5
	9	10	14
7	11	15	3

$\overset{?}{\rightsquigarrow}$

1	2	3	4
5	6	7	8
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13	14	15	

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↑

states:	game configurations
transitions:	legal moves

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  with shared variables $\mathit{field}[i]$ for $i = 1, \ldots, 16$

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CTL specification:

$$\exists\Diamond \bigwedge_{1 \leq i \leq 15} \text{“}\mathit{piece}\ i \text{ on } \mathit{field}[i]\text{”}$$

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CTL specification: seeking for a **witness** for

$$\exists\Diamond \bigwedge_{1 \leq i \leq 15} \text{"}\mathit{piece}\ i \text{ on } \mathit{field}[i]\text{"}$$

Semantics of CTL

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define a satisfaction relation $\models$ for CTL formulas over AP and a given TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$

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- interpretation of state formulas over the states
- interpretation of path formulas over the paths (infinite path fragments)

Recall: semantics of LTL

for infinite path fragment $\pi = s_0\, s_1\, s_2\, \ldots$:

$\pi \models true$

$\pi \models a$ iff $s_0 \models a$,i.e., $a \in L(s_0)$

$\pi \models \varphi_1 \wedge \varphi_2$ iff $\pi \models \varphi_1$ and $\pi \models \varphi_2$

$\pi \models \neg\varphi$ iff $\pi \not\models \varphi$

$\pi \models \bigcirc\varphi$ iff $suffix(\pi, 1) = s_1\, s_2\, s_3 \ldots \models \varphi$

$\pi \models \varphi_1 \, \mathsf{U} \, \varphi_2$ iff there exists $j \geq 0$ such that

$suffix(\pi, j) = s_j\, s_{j+1}\, s_{j+2} \ldots \models \varphi_2$ and

$suffix(\pi, k) = s_k\, s_{k+1}\, s_{k+2} \ldots \models \varphi_1$ for $0 \leq k < j$

Satisfaction relation for path formulas

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semantics of derived operators:

$\pi \models \lozenge\Phi$ iff there exists $j \geq 0$ with $s_j \models \Phi$

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		$s_k \models \Phi_1$ for $0 \leq k < j$

semantics of derived operators:

$\pi \models \Diamond \Phi$	iff	there exists $j \geq 0$ with	$s_j \models \Phi$
$\pi \models \Box \Phi$	iff	for all $j \geq 0$ we have:	$s_j \models \Phi$

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$s \models \neg\Phi$ iff $s \not\models \Phi$

Satisfaction relation for state formulas

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$s \models \exists\varphi$	iff	there is a path $\pi \in \mathit{Paths}(s)$ s.t. $\pi \models \varphi$

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$s \models \forall\varphi$	iff	for each path $\pi \in \mathit{Paths}(s)$: $\pi \models \varphi$

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satisfaction set for state formula Φ:

$$\mathit{Sat}(\Phi) \stackrel{\text{def}}{=} \{s \in S : s \models \Phi\}$$

Interpretation of CTL formulas over a TS

Interpretation of CTL formulas over a TS

satisfaction of state formulas over a TS $\mathcal{T}$:

$$\mathcal{T} \models \Phi \quad \text{iff} \quad S_0 \subseteq Sat(\Phi)$$

where S_0 is the set of initial states

recall: $Sat(\Phi) \; = \; \{ s \in S : s \models \Phi \}$

Interpretation of CTL formulas over a TS

satisfaction of state formulas over a TS $\mathcal{T}$:

$$\mathcal{T} \models \Phi \quad \text{iff} \quad S_0 \subseteq Sat(\Phi)$$
$$\text{iff} \quad s_0 \models \Phi \text{ for all initial states } s_0 \text{ of } \mathcal{T}$$

where S_0 is the set of initial states

recall: $Sat(\Phi) \;=\; \{ s \in S : s \models \Phi \}$

Semantics of the next operator

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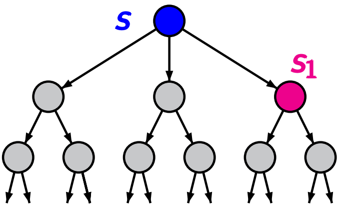
$$s \models \exists\bigcirc\Phi \quad \text{iff} \quad \text{there exists } \pi = s\, s_1\, s_2 \ldots \in Paths(s)$$
$$\text{s.t. } \pi \models \bigcirc\Phi$$

$s \models \exists\bigcirc\Phi$ iff there exists $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$
s.t. $\pi \models \bigcirc\Phi$, i.e., $s_1 \models \Phi$

$\exists\bigcirc\Phi$



$Post(s) \cap Sat(\Phi) \neq \varnothing$

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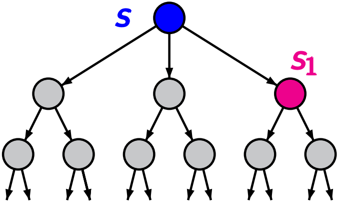
$s \models \forall \bigcirc \Phi$ iff for all $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$: $\pi \models \bigcirc \Phi$

$\exists \bigcirc \Phi$



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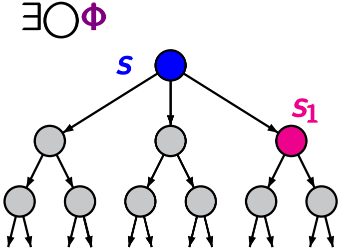
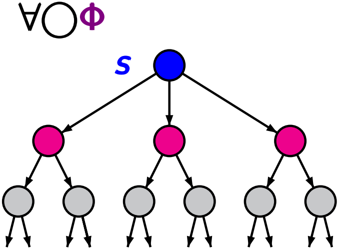
Semantics of the next operator

$s \models \exists \bigcirc \Phi$ iff there exists $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$ s.t. $\pi \models \bigcirc \Phi$, i.e., $s_1 \models \Phi$

$s \models \forall \bigcirc \Phi$ iff for all $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$: $\pi \models \bigcirc \Phi$, i.e., $s_1 \models \Phi$



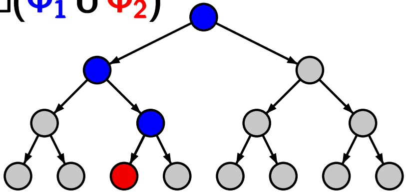
$Post(s) \cap Sat(\Phi) \neq \varnothing$



$Post(s) \subseteq Sat(\Phi)$

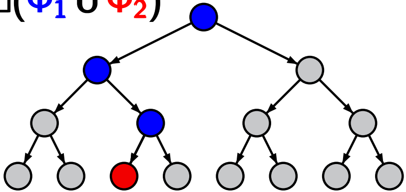
Semantics of until

$\exists(\Phi_1 \, U \, \Phi_2)$

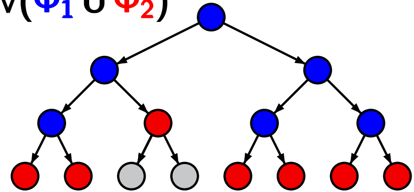


Semantics of until

$\exists(\Phi_1 \mathsf{U} \Phi_2)$

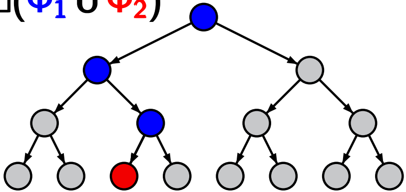


$\forall(\Phi_1 \mathsf{U} \Phi_2)$

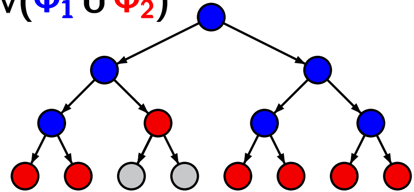


Semantics of until and eventually

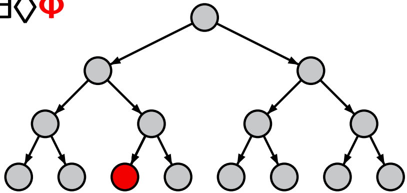
$\exists(\Phi_1 \, \mathsf{U} \, \Phi_2)$



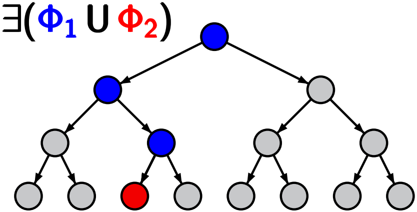
$\forall(\Phi_1 \, \mathsf{U} \, \Phi_2)$

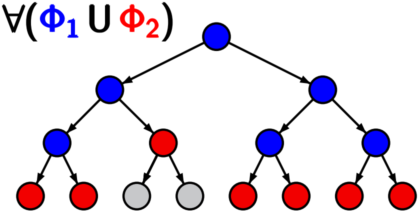


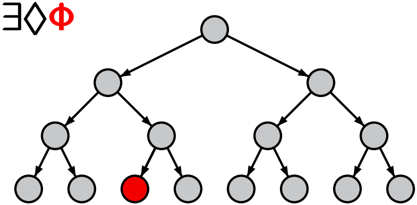
$\exists\Diamond\Phi$

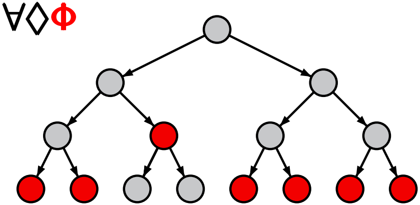


Semantics of until and eventually

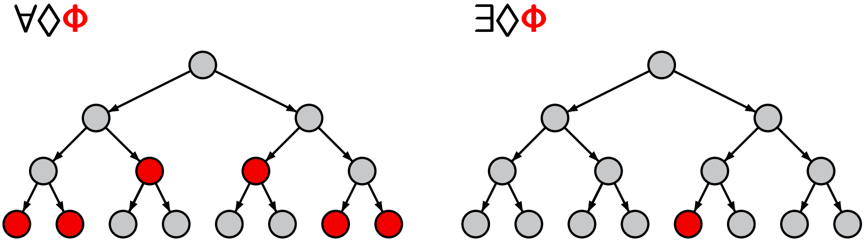






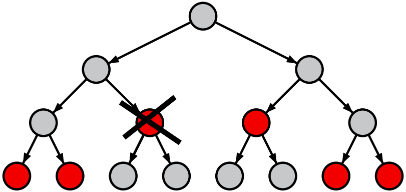


Semantics of eventually and always

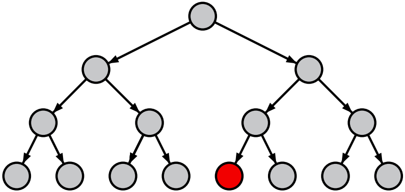


Semantics of eventually and always

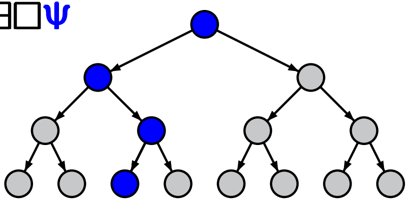
$\neg\forall\Diamond\Phi$



$\exists\Diamond\Phi$

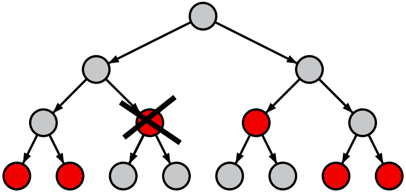


$\exists\Box\Psi$

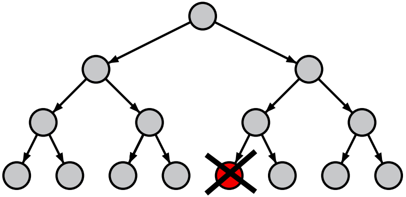


Semantics of eventually and always

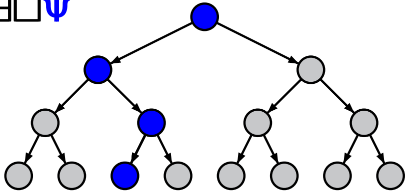
$\neg\forall\Diamond\Phi$



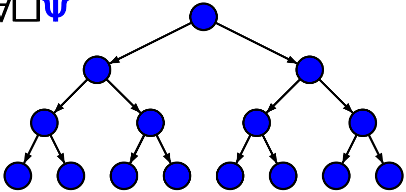
$\neg\exists\Diamond\Phi$



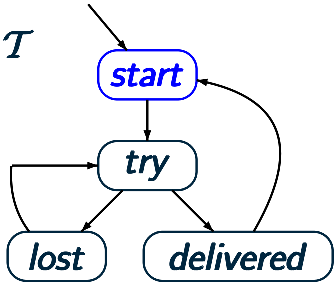
$\exists\Box\Psi$

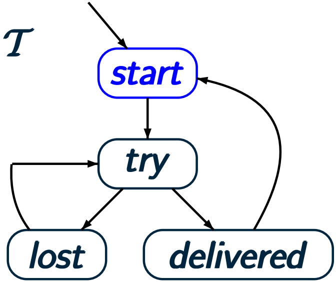


$\forall\Box\Psi$



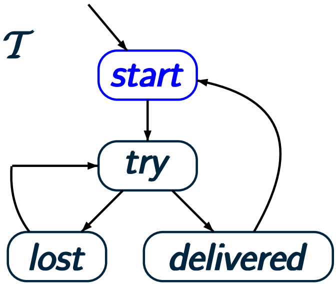
Example for CTL semantics





CTL formula

$\Phi = \forall\Box\forall\Diamond start$

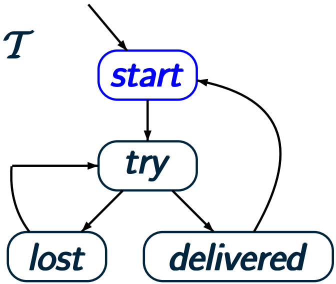


CTL formula

$\Phi = \forall\Box \; \forall\Diamond start$

$Sat(\forall\Diamond start) = \;?$

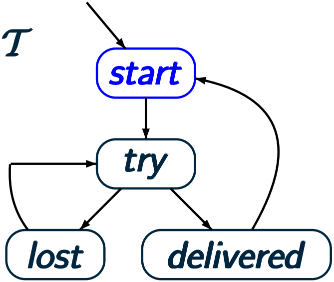
Example for CTL semantics



CTL formula

$\Phi = \forall\Box\, \forall\Diamond start$

$Sat(\forall\Diamond start) = \{start, delivered\}$

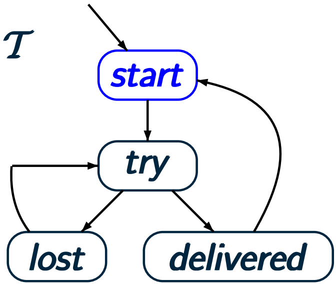


CTL formula

$$\Phi = \forall\Box\forall\Diamond \mathit{start} \quad \widehat{=} \quad \forall\Box(\mathit{start} \vee \mathit{delivered})$$

$$\mathit{Sat}(\forall\Diamond \mathit{start}) = \{\mathit{start}, \mathit{delivered}\}$$

Example for CTL semantics

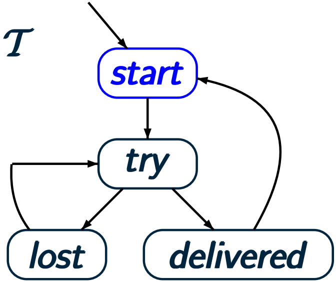


CTL formula

$\Phi = \forall\Box\forall\Diamond start \quad \widehat{=} \quad \forall\Box(start \vee delivered)$

$Sat(\forall\Diamond start) = \{start, delivered\}$

$Sat(\Phi) = \varnothing$



$\mathcal{T} \not\models \forall\Box\forall\Diamond start$

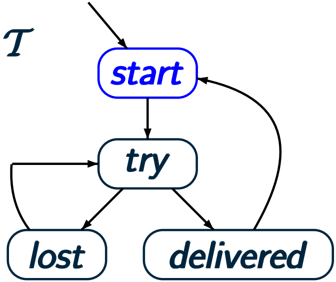
CTL formula

$\Phi = \forall\Box\forall\Diamond start \quad \widehat{=} \quad \forall\Box(start \vee delivered)$

$$Sat(\forall\Diamond start) = \{start, delivered\}$$

$$Sat(\Phi) = \varnothing$$

Example for CTL semantics



$\mathcal{T} \not\models \forall\Box\forall\Diamond start$

"infinitely often *start*"

CTL formula

$\Phi = \forall\Box\forall\Diamond start \ \widehat{=}\ \forall\Box(start \vee delivered)$

$Sat(\forall\Diamond start) = \{start, delivered\}$

$Sat(\Phi) = \varnothing$

Specifying "infinitely often" in CTL

Specifying "infinitely often" in CTL

If s is a state in a TS and $a \in AP$ then:

$$s \models_{\text{CTL}} \forall\Box\forall\Diamond a$$

iff for all paths $\pi = s_0\, s_1\, s_2 \ldots \in Paths(s)$:

$$\overset{\infty}{\exists}\, i \geq 0. \quad \text{s.t.} \quad s_i \models a$$

If s is a state in a TS and $a \in AP$ then:

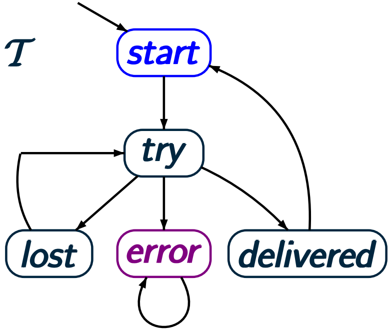
$$s \models_{\text{CTL}} \forall\Box\forall\Diamond a$$

iff for all paths $\pi = s_0\, s_1\, s_2 \ldots \in Paths(s)$:

$$\overset{\infty}{\exists}\, i \geq 0. \quad \text{s.t.} \quad s_i \models a$$

iff $s \models_{\text{LTL}} \Box\Diamond a$

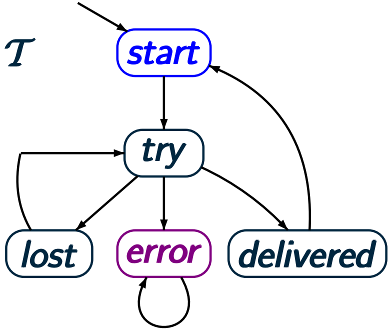
Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond\forall\Box\neg start$

Example: CTL semantics

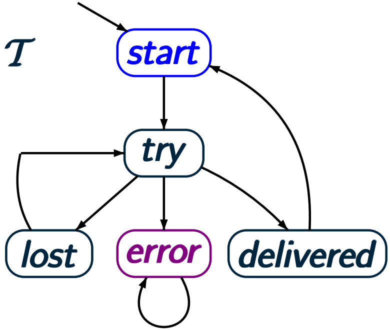


$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



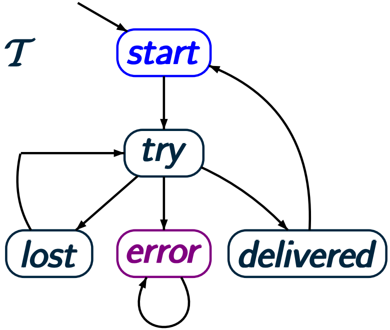
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) =$?

Example: CTL semantics



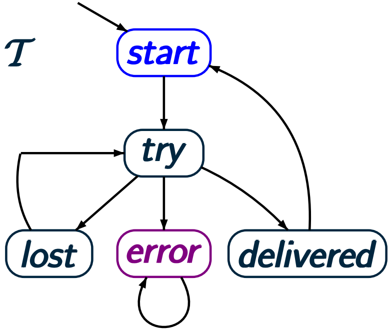
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$ ✓

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) = Sat(\exists\Diamond error) =$ "all states"

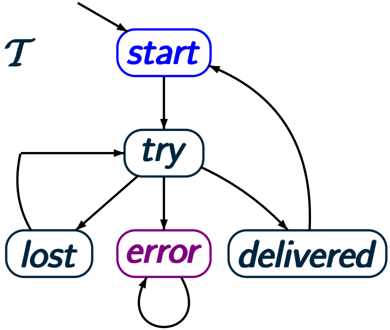
Example: CTL semantics



$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**



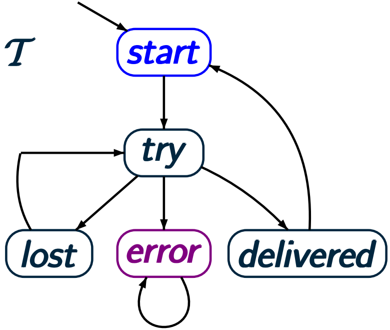
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**

$\Phi_2 = \forall\bigcirc\exists\bigcirc \boxed{\forall\Box\neg start}$

$$Sat(\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

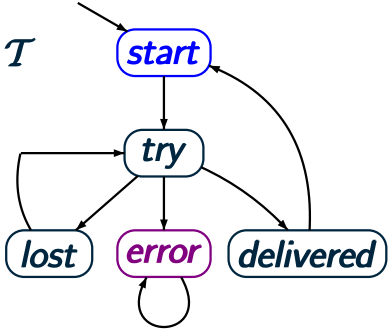
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**

$\Phi_2 = \forall\bigcirc \boxed{\exists\bigcirc\forall\Box\neg start} \rightsquigarrow \forall\bigcirc \boxed{\exists\bigcirc error}$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$?

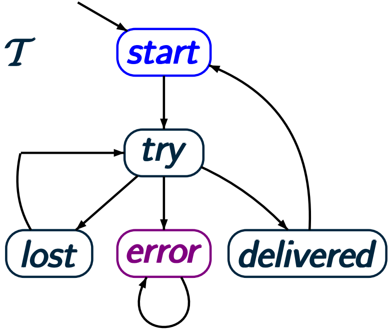
$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start \quad \rightsquigarrow \quad \forall\bigcirc(error \vee try)$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$

$$Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) = \ ?$$

Example: CTL semantics



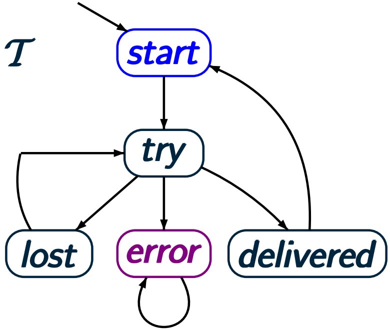
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ $\checkmark$

$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start \rightsquigarrow \forall\bigcirc(error \vee try)$

$$Sat(\forall\Box\neg start) = \{error\}$$
$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$
$$Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) = \{error, lost, start\}$$

Example: CTL semantics



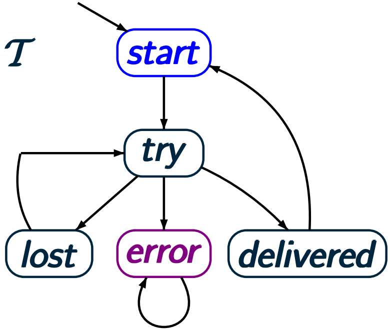
$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

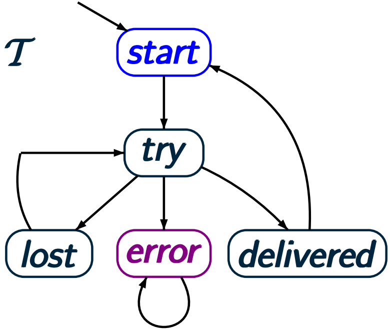
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$?

$\Phi_3 = \exists\bigcirc\forall\bigcirc \boxed{\forall\Box\neg start}$

$Sat(\forall\Box\neg start) = \{error\}$

Example: CTL semantics



$\mathcal{T} \models \exists\lozenge\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

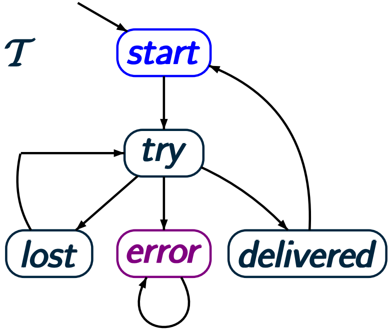
$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

$$\Phi_3 = \exists\bigcirc \boxed{\forall\bigcirc\forall\Box\neg start} \rightsquigarrow \exists\bigcirc \boxed{\forall\bigcirc error}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\forall\bigcirc\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

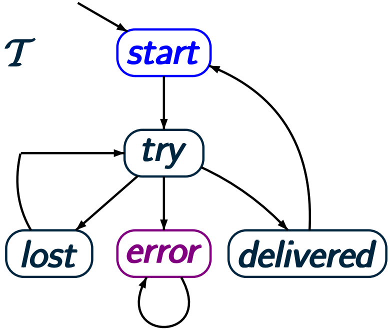
$$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start \rightsquigarrow \boxed{\exists\bigcirc error}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\forall\bigcirc\forall\Box\neg start) = \{error\}$$

$$Sat(\exists\bigcirc\forall\bigcirc\forall\Box\neg start) = \ ?$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

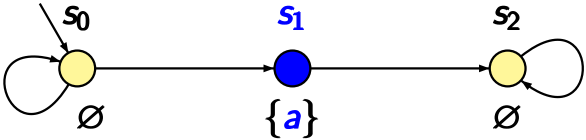
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \not\models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$

$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start \rightsquigarrow$ $\boxed{\exists\bigcirc error}$

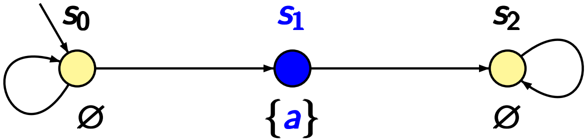
$$
\begin{aligned}
Sat(\forall\Box\neg start) &= \{error\} \\
Sat(\forall\bigcirc\forall\Box\neg start) &= \{error\} \\
Sat(\exists\bigcirc\forall\bigcirc\forall\Box\neg start) &= \{error, try\}
\end{aligned}
$$

Example: CTL semantics



does $\mathcal{T} \models \exists \bigcirc \forall \Box \neg a$ hold ?

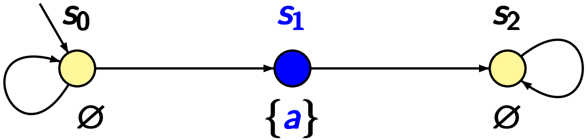
Example: CTL semantics



does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold **?**

answer: no

Example: CTL semantics

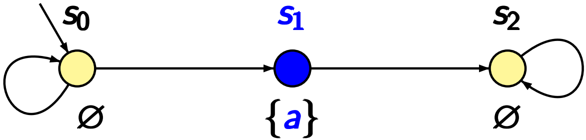


does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

$$Sat(\forall\Box\neg a) \;=\; \{s_2\}$$

Example: CTL semantics

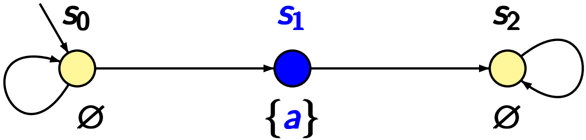


does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

$$Sat(\forall\Box\neg a) = \{s_2\}$$
$$Sat(\exists\bigcirc\forall\Box\neg a) = \{s_2, s_1\}$$

Example: CTL semantics

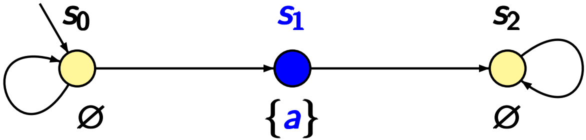


does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

does $\mathcal{T} \models \forall\Box\exists\bigcirc\neg a$ hold ?

Example: CTL semantics



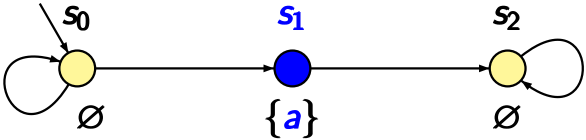
does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

does $\mathcal{T} \models \forall\Box\exists\bigcirc\neg a$ hold ?

answer: yes

Example: CTL semantics



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$$Sat(\exists\bigcirc\neg a) = \{s_0, s_1, s_2\}$$

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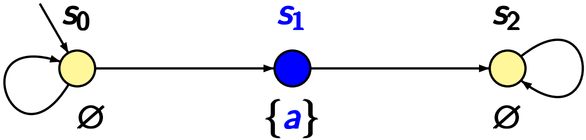

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\end{aligned}
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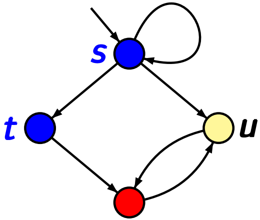

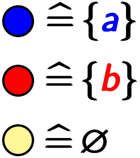

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$?

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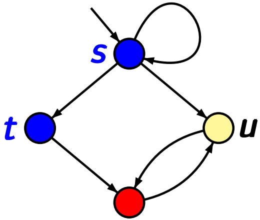

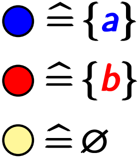

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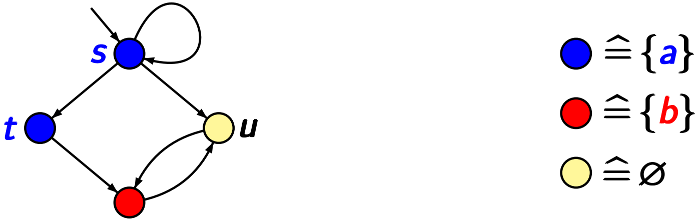

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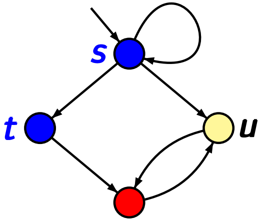

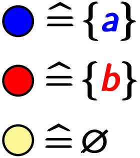

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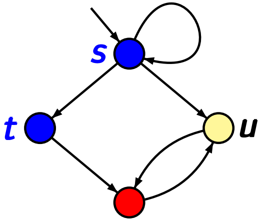

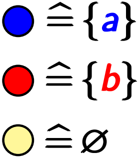

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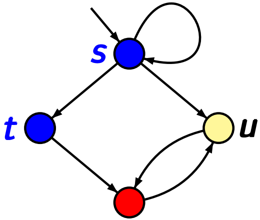

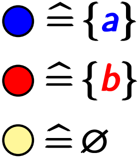

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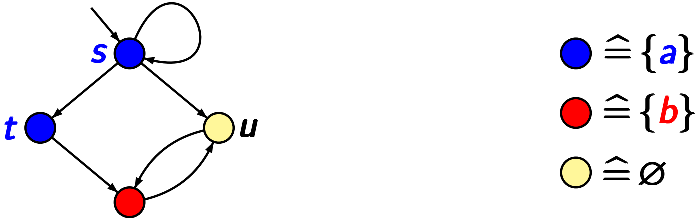

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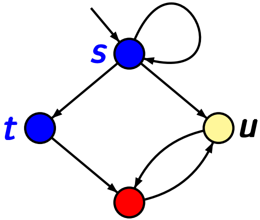

- $\mathrel{\widehat{=}} \{a\}$ (blue)
- $\mathrel{\widehat{=}} \{b\}$ (red)
- $\mathrel{\widehat{=}} \varnothing$ (yellow)

$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$ ✓ as $s\,s\,s \ldots \models \Box\exists(a \,\mathsf{U}\, b)$

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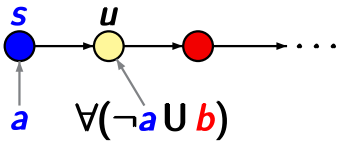

$\models a \,\mathsf{U}\, \forall(\neg a \,\mathsf{U}\, b)$

Correct or wrong?

Let $\mathcal{T}$ be a transition system and Φ a CTL formula.
Is the following statement correct **?**

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	iff	there exists an initial state s_0 with $s_0 \models \Phi$

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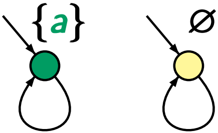
transition system $\mathcal{T}$ with 2 initial states:



$\mathcal{T} \not\models \exists\Box a$

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Hamilton path problem

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question: does G have a Hamilton path, i.e., a path that visits each node exactly once ?

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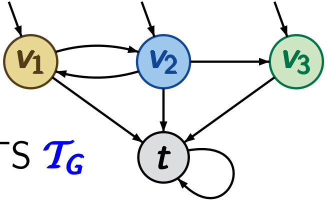
digraph G

$\leadsto$



TS $\mathcal{T}_G$

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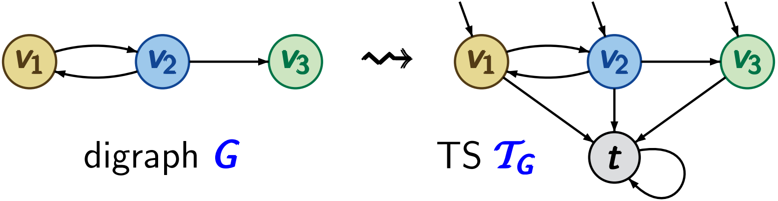

CTL formula Φ

$$(v_1 \wedge \exists\bigcirc(v_2 \wedge \exists\bigcirc v_3)) \vee (v_1 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_2)) \vee$$
$$(v_2 \wedge \exists\bigcirc(v_1 \wedge \exists\bigcirc v_3)) \vee (v_2 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_1)) \vee$$
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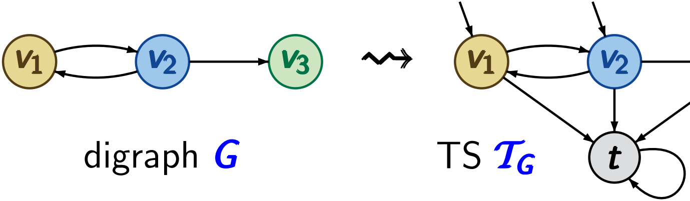

digraph G $\rightsquigarrow$ TS $\mathcal{T}_G$

CTL formula Φ = negation of the formula

$$(v_1 \wedge \exists\bigcirc(v_2 \wedge \exists\bigcirc v_3)) \vee (v_1 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_2)) \vee$$
$$(v_2 \wedge \exists\bigcirc(v_1 \wedge \exists\bigcirc v_3)) \vee (v_2 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_1)) \vee$$
$$(v_3 \wedge \exists\bigcirc(v_1 \wedge \exists\bigcirc v_2)) \vee (v_3 \wedge \exists\bigcirc(v_2 \wedge \exists\bigcirc v_1))$$

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$\Phi_1 \equiv \Phi_2$ iff for all transition systems $\mathcal{T}$:

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quantification over all transition systems $\mathcal{T}$

- without terminal states
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Examples:

$$\neg\neg\Phi \equiv \Phi$$

$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$

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$$\neg\neg\Phi \equiv \Phi$$
$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$
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Correct or wrong?

$$\exists\Diamond(a \wedge b) \equiv \exists\Diamond a \wedge \exists\Diamond b$$

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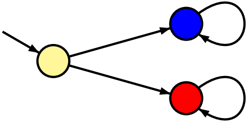

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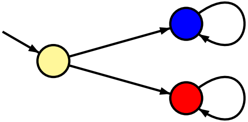

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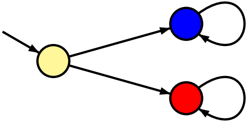

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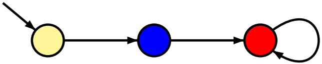

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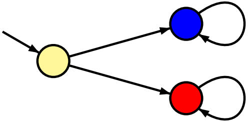

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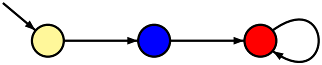

but:

$$\forall\Box(\Phi_1 \wedge \Phi_2) \;\equiv\; \forall\Box\Phi_1 \;\wedge\; \forall\Box\Phi_2$$
$$\exists\Diamond(\Phi_1 \vee \Phi_2) \;\equiv\; \exists\Diamond\Phi_1 \;\vee\; \exists\Diamond\Phi_2$$

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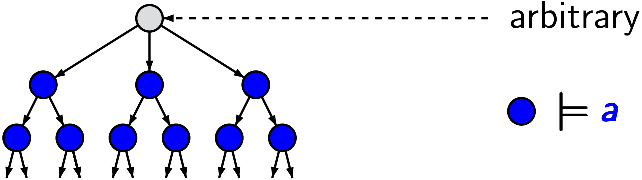
both formulas require computation trees of the form:



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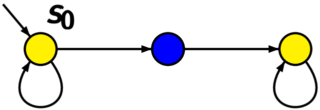

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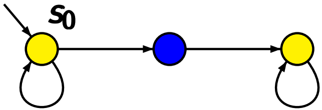

$s_0 \not\models \exists\bigcirc\exists\Box a$

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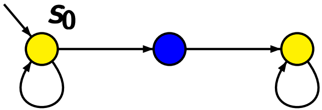

$s_0 \not\models \exists\bigcirc\exists\Box a$

note: $Sat(\exists\Box a) = \varnothing$

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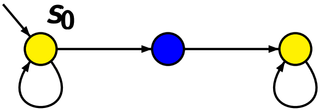

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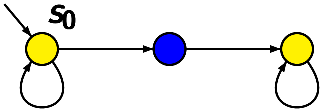

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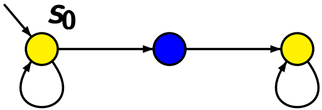

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$s_0 \models \exists\bigcirc a$

$\Longrightarrow\ s_0\, s_0\, s_0 \ldots \models \Box\exists\bigcirc a$

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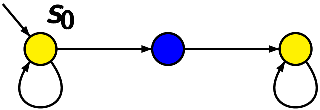
$\forall\bigcirc\forall\Box a \equiv \forall\Box\forall\bigcirc a$

correct.

$\exists\bigcirc\exists\Box a \equiv \exists\Box\exists\bigcirc a$

wrong, e.g.,



$s_0 \not\models \exists\bigcirc\exists\Box a$

$s_0 \models \exists\Box\exists\bigcirc a$

$s_0 \models \exists\bigcirc a$

$\Longrightarrow \quad s_0\, s_0\, s_0 \ldots \models \Box\exists\bigcirc a$

$\Longrightarrow \quad s_0 \models \exists\Box\exists\bigcirc a$

Weak until W

Weak until W

in **LTL**: $\varphi \, \mathsf{W} \, \psi \overset{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \lor \Box\varphi$

in **CTL**: **?**

Weak until W

in **LTL**: $\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

in **CTL**: **?**

in **LTL**: $\varphi \, \mathsf{W} \, \psi \overset{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\forall((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

in **LTL**: $\varphi \, \mathsf{W} \, \psi \overset{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\forall((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\exists((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

Weak until W in CTL

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\forall\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\exists\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi\, \mathsf{W}\, \Psi) \stackrel{\text{def}}{=} \neg\forall(\,(\Phi \wedge \neg\Psi)\, \mathsf{U}\, (\neg\Phi \wedge \neg\Psi)\,)$$

$$\forall(\Phi\, \mathsf{W}\, \Psi) \stackrel{\text{def}}{=} \neg\exists(\,(\Phi \wedge \neg\Psi)\, \mathsf{U}\, (\neg\Phi \wedge \neg\Psi)\,)$$

note that:

$$\exists(\Phi\, \mathsf{W}\, \Psi) \equiv \exists(\Phi\, \mathsf{U}\, \Psi) \vee \exists\Box\Phi$$

Weak until W in CTL

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\forall\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

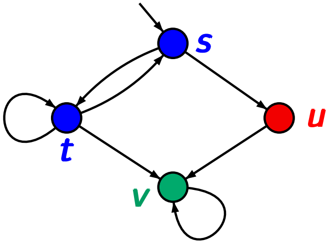
$$\forall(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\exists\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

note that:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \exists(\Phi \, \mathsf{U} \, \Psi) \vee \exists\Box\Phi$$

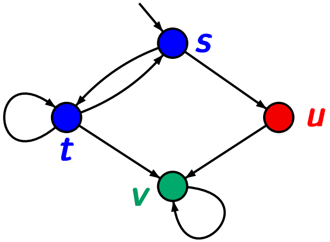
$$\forall(\Phi \, \mathsf{W} \, \Psi) \not\equiv \forall(\Phi \, \mathsf{U} \, \Psi) \vee \forall\Box\Phi$$

Weak until W in CTL



$\bullet \mathrel{\widehat{=}} \{a\}$
$\bullet \mathrel{\widehat{=}} \{b\}$
$\bullet \mathrel{\widehat{=}} \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W}\, c)$ **?**



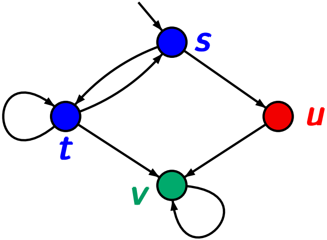
$\bullet \;\widehat{=}\; \{a\}$

$\bullet \;\widehat{=}\; \{b\}$

$\bullet \;\widehat{=}\; \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ $\surd$ as $s \models \exists(a \,\mathsf{W}\, c)$

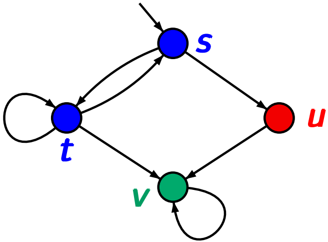
Weak until W in CTL



● $\widehat{=}$ $\{a\}$
● $\widehat{=}$ $\{b\}$
● $\widehat{=}$ $\{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \; \mathsf{W} \; c)$ $\surd$ as $s\, s_1\, s_2 \ldots \models \Diamond\exists(a \; \mathsf{W} \; c)$

Weak until W in CTL



$\bullet \;\widehat{=}\; \{a\}$

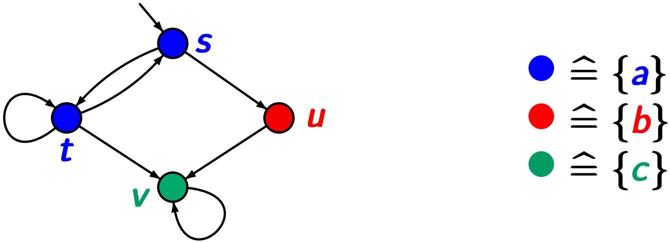
$\bullet \;\widehat{=}\; \{b\}$

$\bullet \;\widehat{=}\; \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ $\surd$ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ **?**

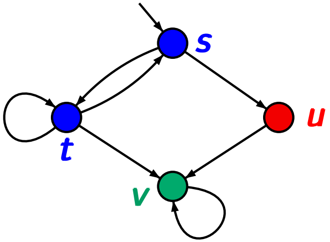
Weak until W in CTL



$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ $\surd$ as $s \models \exists(a \mathsf{W} c)$

$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ $\surd$ as $s \models \exists\Box a$

Weak until W in CTL



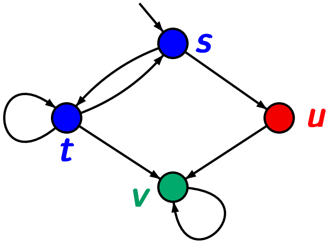
● $\widehat{=}$ $\{a\}$
● $\widehat{=}$ $\{b\}$
● $\widehat{=}$ $\{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \, \mathsf{W} \, c)$ √ as $s \models \exists(a \, \mathsf{W} \, c)$

$\mathcal{T} \models \exists(a \, \mathsf{W} \, \exists\Box b)$ √ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \, \mathsf{W} \, (a \wedge b))$?

Weak until W in CTL



$\bullet \;\widehat{=}\; \{a\}$

$\bullet \;\widehat{=}\; \{b\}$

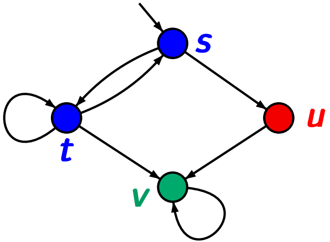
$\bullet \;\widehat{=}\; \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ $\surd$ as $s \models \exists(a \mathsf{W} c)$

$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ $\surd$ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathsf{W} (a \wedge b))$ $\surd$

Weak until W in CTL



$\bullet \mathrel{\widehat{=}} \{a\}$ (blue)
$\bullet \mathrel{\widehat{=}} \{b\}$ (red)
$\bullet \mathrel{\widehat{=}} \{c\}$ (green)

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ ✓ as $s \models \exists(a \,\mathsf{W}\, c)$

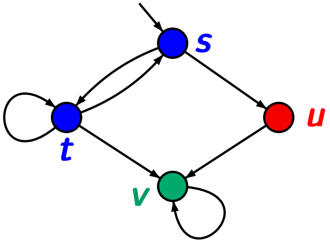
$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ ✓ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$ ✓

↑

three types of paths: $(s\,t)^\omega$ or $(s\,t)^+\,v^\omega$ or $(s\,t)^*\,s\,u\,v^\omega$

Weak until W in CTL



● $\hat{=} \{a\}$
● $\hat{=} \{b\}$
● $\hat{=} \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ ✓ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ ✓ as $s \models \exists\Box a$

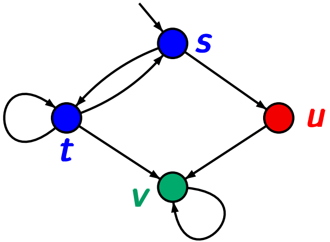
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$ ✓

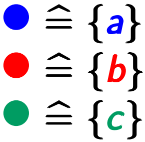
↑

three types of paths: $(s\,t)^\omega$ or $(s\,t)^+\, v^\omega$ or $(s\,t)^*\, s\, u\, v^\omega$

in all three cases: $\pi \models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

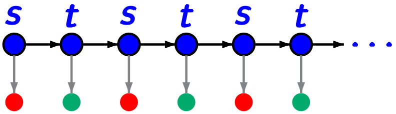




$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ √ as $s \models \exists(a \mathsf{W} c)$

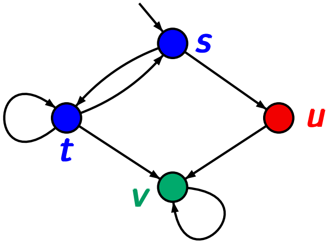
$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ √ as $s \models \exists\Box a$

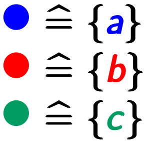
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathsf{W} (a \wedge b))$ √



$\models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

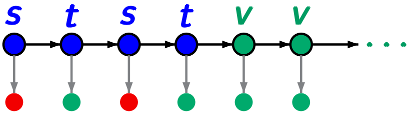




$\mathcal{T} \models \forall\Diamond\exists(a \mathbin{\mathsf{W}} c)$ ✓ as $s \models \exists(a \mathbin{\mathsf{W}} c)$

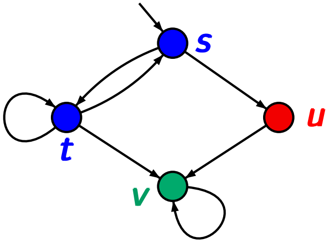
$\mathcal{T} \models \exists(a \mathbin{\mathsf{W}} \exists\Box b)$ ✓ as $s \models \exists\Box a$

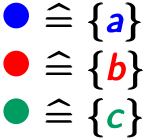
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathbin{\mathsf{W}} (a \wedge b))$ ✓



$\models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

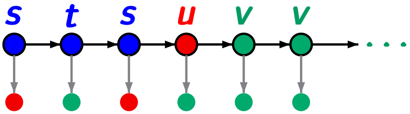




$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ √ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ √ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$ √



$\models \Box\exists\bigcirc(b \vee c)$

Expansion laws

CTLSS4.1-26

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv ?$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \;\equiv\; \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \;\equiv\; \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \;\equiv\; \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \;\equiv\; \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \;\equiv$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{W}\, \Psi))$$

$$\forall(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{W}\, \Psi))$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{W} \, \Psi))$$

$$\exists\Box\Phi \equiv ?$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{W}\, \Psi))$$

$$\forall(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{W}\, \Psi))$$

$$\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{W} \, \Psi))$$

$$\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$$

$$\forall\Box\Phi \equiv \Phi \wedge \forall\bigcirc\forall\Box\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi \, \mathsf{U} \, \Psi)$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi \,\mathbf{U}\, \Psi) \equiv \neg\exists((\Phi \wedge \neg\Psi) \,\mathbf{W}\, (\neg\Phi \wedge \neg\Psi))$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \neg\exists((\Phi \wedge \neg\Psi) \,\mathsf{W}\, (\neg\Phi \wedge \neg\Psi))$$

$$\equiv \neg\exists((\neg\Psi) \,\mathsf{W}\, (\neg\Phi \wedge \neg\Psi))$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\begin{aligned}\forall(\Phi\,\mathsf{U}\,\Psi) &\equiv \neg\exists((\Phi\wedge\neg\Psi)\,\mathsf{W}\,(\neg\Phi\wedge\neg\Psi))\\ &\equiv \neg\exists((\neg\Psi)\,\mathsf{W}\,(\neg\Phi\wedge\neg\Psi))\\ &\equiv \neg\exists((\neg\Psi)\,\mathsf{U}(\neg\Phi\wedge\neg\Psi)) \;\wedge\; \neg\exists\Box\neg\Psi\end{aligned}$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W** yields

$$\forall(\Phi\,\mathrm{U}\,\Psi) \equiv \neg\exists((\neg\Psi)\,\mathrm{U}(\neg\Phi\wedge\neg\Psi)) \wedge \neg\exists\Box\neg\Psi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

derivation of $\forall$U from $\exists$U and $\exists\Box$:

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \neg\exists((\neg\Psi)\,\mathsf{U}(\neg\Phi \wedge \neg\Psi)) \wedge \neg\exists\Box\neg\Psi$$

$\forall$U and $\forall\bigcirc$ are expressible via $\exists$U, $\exists\bigcirc$ and $\exists\Box$

Existential normalform for CTL

For each **CTL** formula Ψ there is an equivalent **CTL** formula Φ built by

- operators of propositional logic
- the modalities $\exists\bigcirc$, $\exists \mathbf{U}$ and $\exists\Box$.

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$$\Phi \ ::= \ \mathit{true} \ \big| \ a \ \big| \ \Phi_1 \wedge \Phi_2 \ \big| \ \neg\Phi \ \big| \ \exists\bigcirc\Phi \ \big| \ \exists(\Phi_1 \, \mathsf{U} \, \Phi_2) \ \big| \ \exists\Box\Phi$$

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$$\Phi ::= true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\bigcirc\Phi \mid \exists(\Phi_1 \mathrm{U} \Phi_2) \mid \exists\Box\Phi$$

transformation $\Psi \rightsquigarrow \Phi$ relies on:

$$\forall\bigcirc\Psi \rightsquigarrow \neg\exists\bigcirc\neg\Psi$$

$$\forall(\Psi_1 \mathrm{U} \Psi_2) \rightsquigarrow \neg\exists(\neg\Psi_2 \mathrm{U}(\neg\Psi_1 \wedge \neg\Psi_2)) \wedge \neg\exists\Box\neg\Psi_2$$

Example: ∃-normal form for CTL formula

CTL formula	$\rightsquigarrow$	CTL formula in ∃-normal form
$\forall \bigcirc \Psi$	$\rightsquigarrow$	$\neg \exists \bigcirc \neg \Psi$
$\forall (\Psi_1 \, \mathsf{U} \, \Psi_2)$	$\rightsquigarrow$	$\neg \exists (\neg \Psi_2 \, \mathsf{U} (\neg \Psi_1 \wedge \neg \Psi_2)) \wedge \neg \exists \Box \neg \Psi_2$

$$\forall ((\forall \bigcirc a) \, \mathsf{U} \, \neg c)$$

Example: $\exists$-normal form for CTL formula

CTL formula	$\rightsquigarrow$	CTL formula in $\exists$-normal form
$\forall\bigcirc\Psi$	$\rightsquigarrow$	$\neg\exists\bigcirc\neg\Psi$
$\forall(\Psi_1 \mathsf{U} \Psi_2)$	$\rightsquigarrow$	$\neg\exists(\neg\Psi_2 \mathsf{U}(\neg\Psi_1 \wedge \neg\Psi_2)) \wedge \neg\exists\Box\neg\Psi_2$

$$\forall((\forall\bigcirc a) \mathsf{U} \neg c)$$

$$\equiv \quad \neg\exists(\neg\neg c \, \mathsf{U}((\neg\forall\bigcirc a) \wedge \neg\neg c)) \quad \wedge \quad \neg\exists\Box\neg\neg c$$

Example: ∃-normal form for CTL formula

CTL formula	$\rightsquigarrow$	CTL formula in ∃-normal form
$\forall\bigcirc\Psi$	$\rightsquigarrow$	$\neg\exists\bigcirc\neg\Psi$
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$$\forall((\forall\bigcirc a) \,\mathsf{U}\, \neg c)$$

$$\equiv \quad \neg\exists(\neg\neg c \,\mathsf{U}((\neg\forall\bigcirc a) \wedge \neg\neg c)) \quad \wedge \quad \neg\exists\Box\neg\neg c$$

$$\equiv \quad \neg\exists(c \,\mathsf{U}((\neg\forall\bigcirc a) \wedge c)) \quad \wedge \quad \neg\exists\Box c$$

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$\forall\bigcirc\Psi$	$\rightsquigarrow$	$\neg\exists\bigcirc\neg\Psi$
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$$\equiv \quad \neg\exists(\neg\neg c \,\mathsf{U}((\neg\forall\bigcirc a) \wedge \neg\neg c)) \;\;\wedge\;\; \neg\exists\Box\neg\neg c$$

$$\equiv \quad \neg\exists(c \,\mathsf{U}((\neg\forall\bigcirc a) \wedge c)) \;\;\wedge\;\; \neg\exists\Box c$$

$$\equiv \quad \neg\exists(c \,\mathsf{U}((\exists\bigcirc\neg a) \wedge c)) \;\;\wedge\;\; \neg\exists\Box c$$

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but no additional operator for $\bigcirc$ required syntax of **CTL** formulas in **PNF**:

state formulas:

$$\Phi \ ::= \ \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid \exists\varphi \mid \forall\varphi$$

path formulas:

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path formulas:

$$\varphi ::= \bigcirc \Phi \mid \Phi_1 \, \mathsf{U} \, \Phi_2 \mid \Phi_1 \, \mathsf{W} \, \Phi_2$$

For each **CTL** formula Ψ there is an equivalent **CTL** formula Φ in **PNF**

CTL formula ⇝ CTL formula in PNF

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$\neg \mathit{true} \leadsto \mathit{false}$

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... exponential blowup possible ...