# General Picture of LTL Model Checking with Büchi Automata

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## Topics

- Automata-based LTL model checking. General picture.
- From LTL formulas to NBAs. Examples.
- NFA and NBA for safety properties.
- Examples of LTL model checking with NBA.
- LTL model checking complexity (without proof).

## Material

Reading:

Chapter 5 of the book, pages 225-243.

More:

The slides in the following pages are taken from the material of the course "Introduction to Model Checking" held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

# Overview

# LTL model checking problem

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(without terminal states)
LTL-formula $\varphi$ over $AP$

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$$\pi \not\models \varphi, \text{ i.e., } \pi \models \neg\varphi$$

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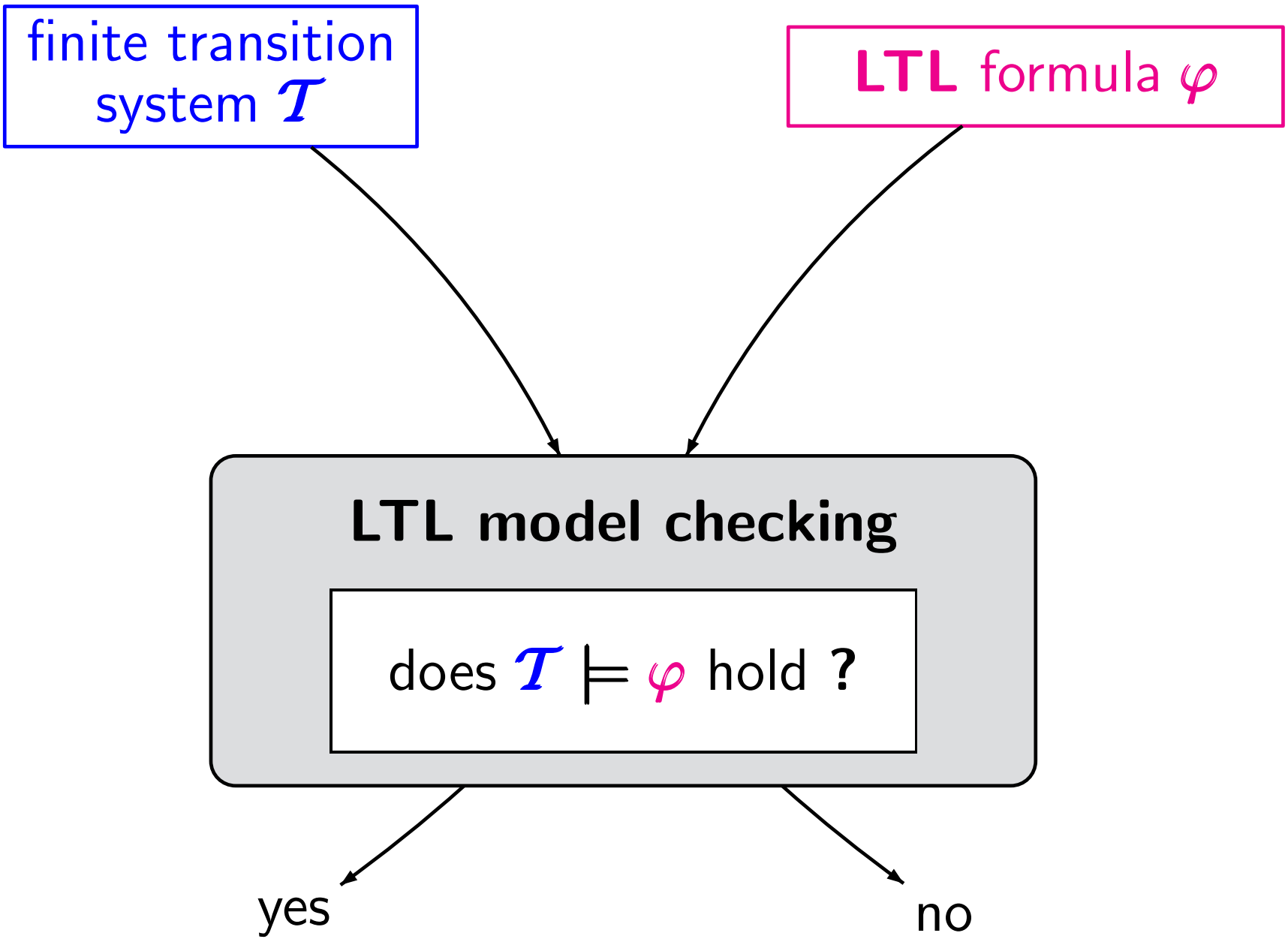
construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets the acceptance condition of $\mathcal{A}$

# Automata-based LTL model checking





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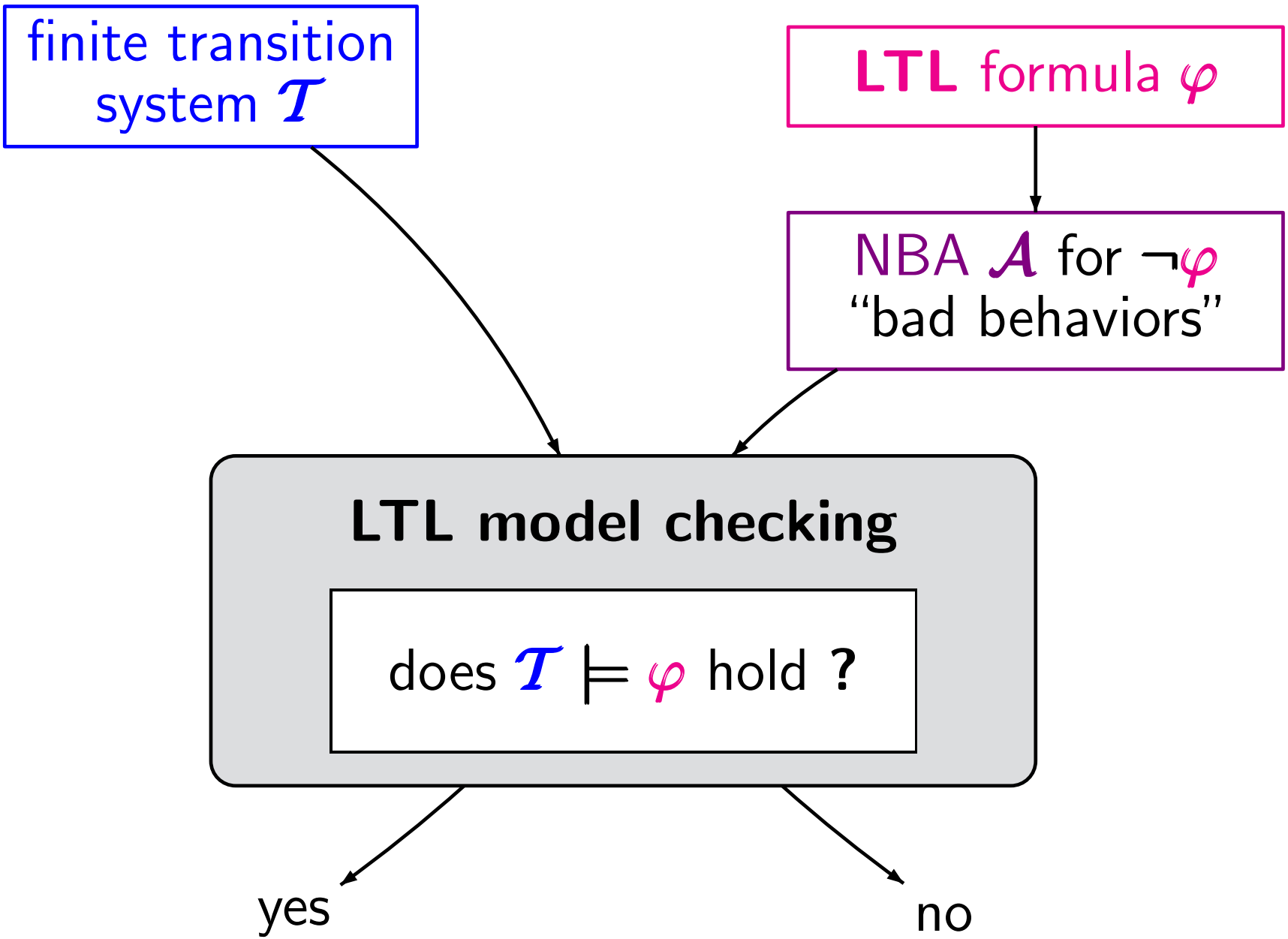

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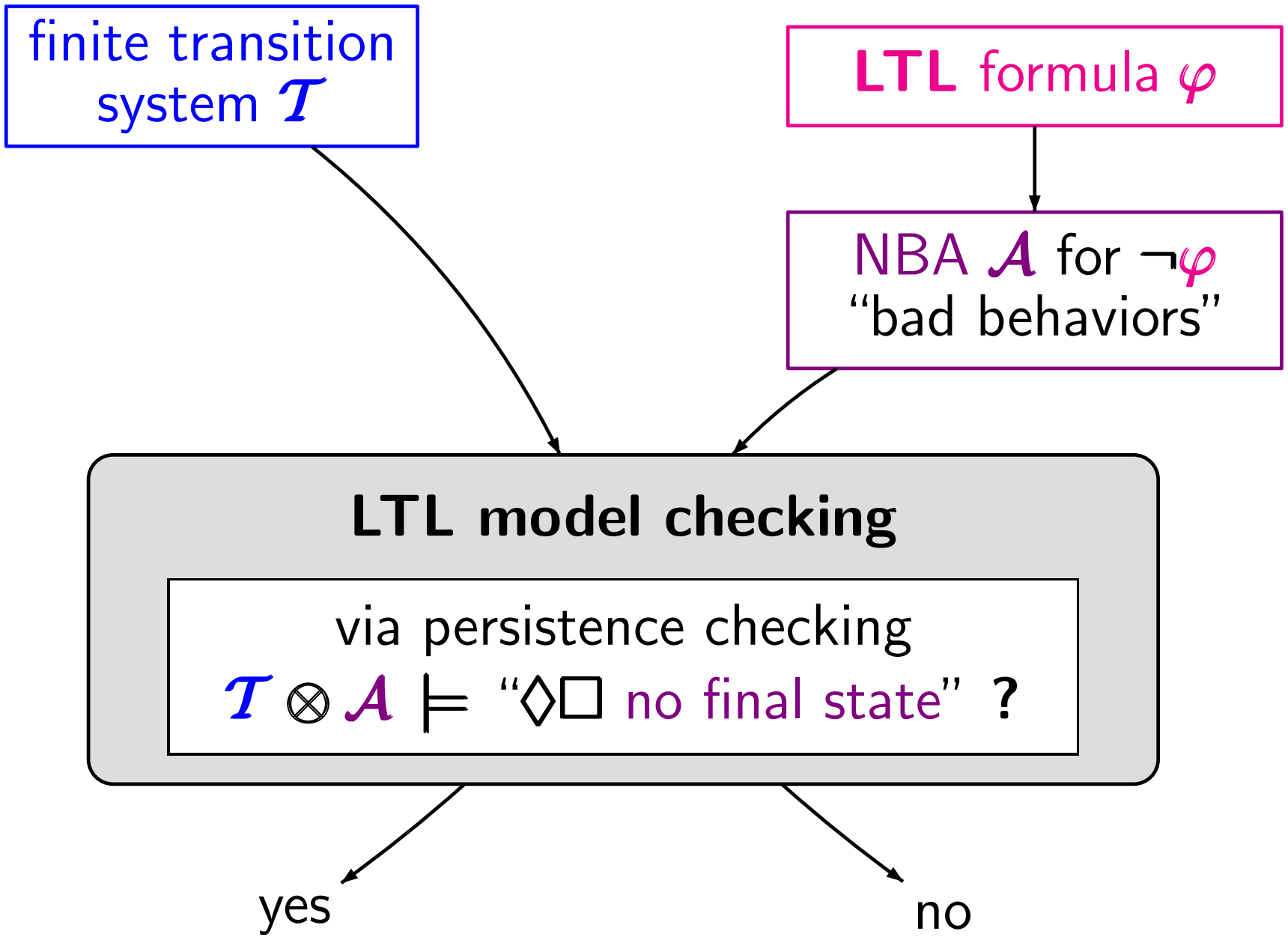

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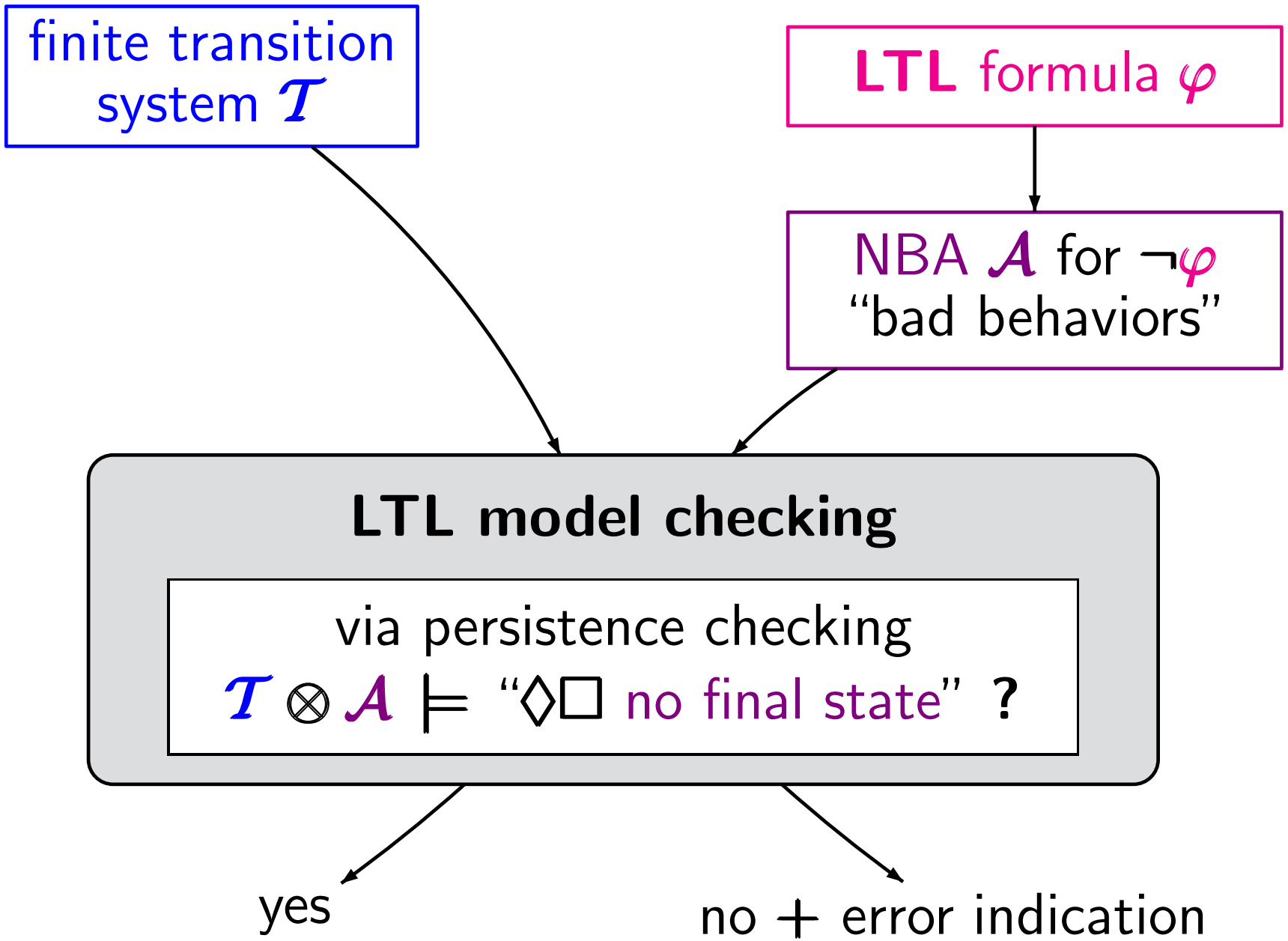

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$\mathcal{T} \models$ safety property $E$

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where $\mathcal{A}$ is an NFA for the bad prefixes

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iff $\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$ $\longleftarrow$ persistence checking

# Recall: nondeterministic Büchi automata



NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- $Q$ finite set of states
- $\Sigma$ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
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run for a word $A_0 A_1 A_2 \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0 q_1 q_2 \ldots$ where $q_0 \in Q_0$ and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run $\pi$ is accepting if $\overset{\infty}{\exists}\, i \in \mathbb{N}.\ q_i \in F$

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*proof:* ... later ...

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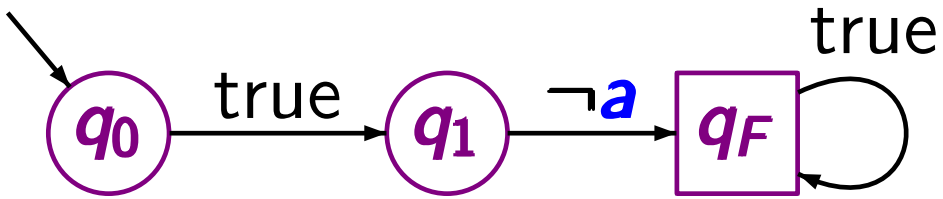


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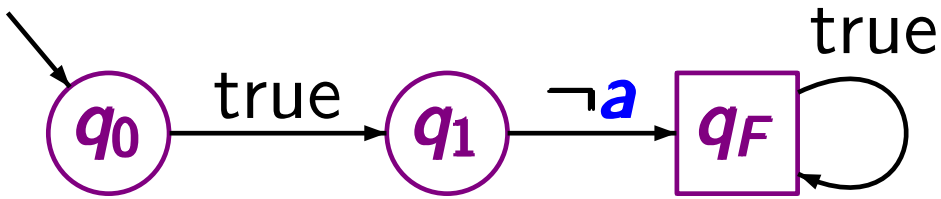


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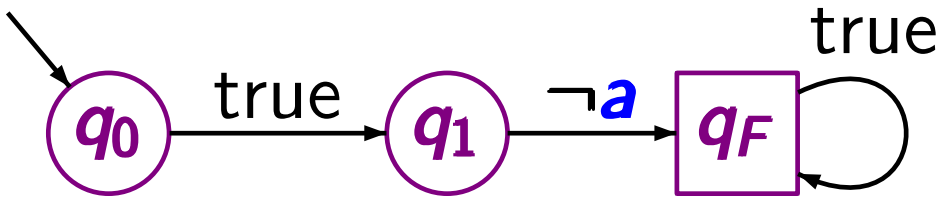


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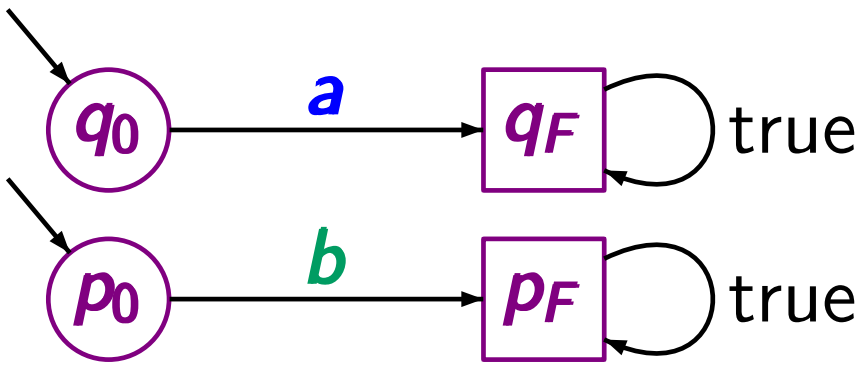


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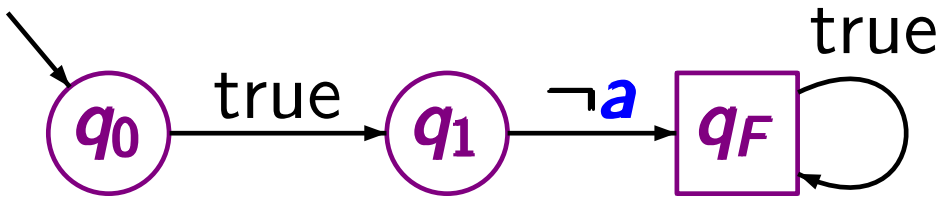


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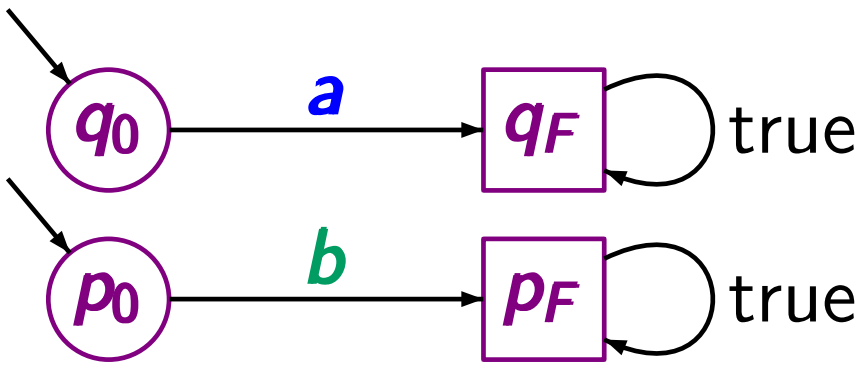


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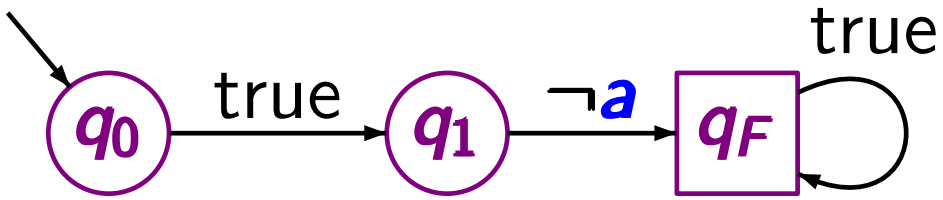


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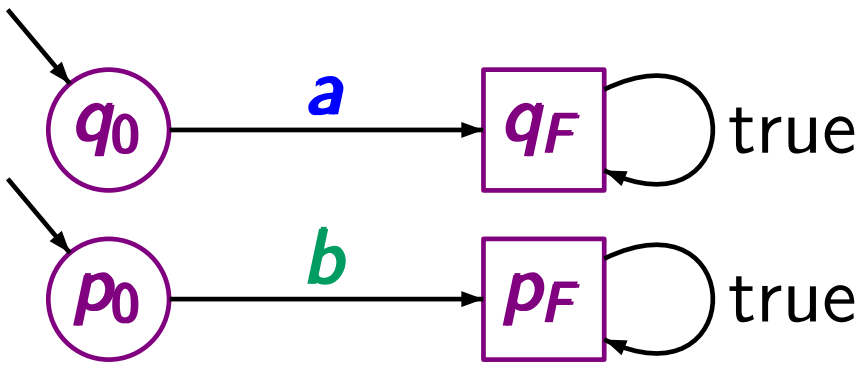


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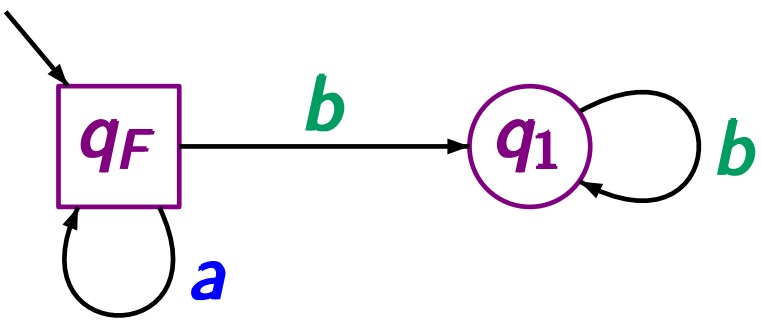


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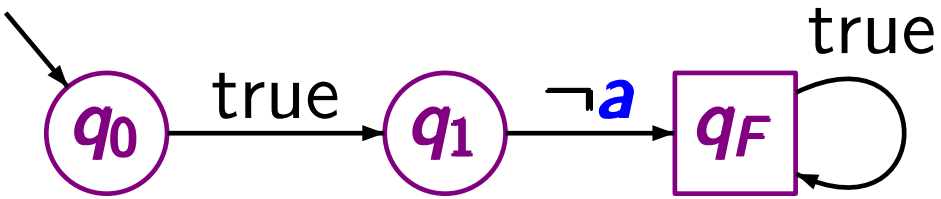


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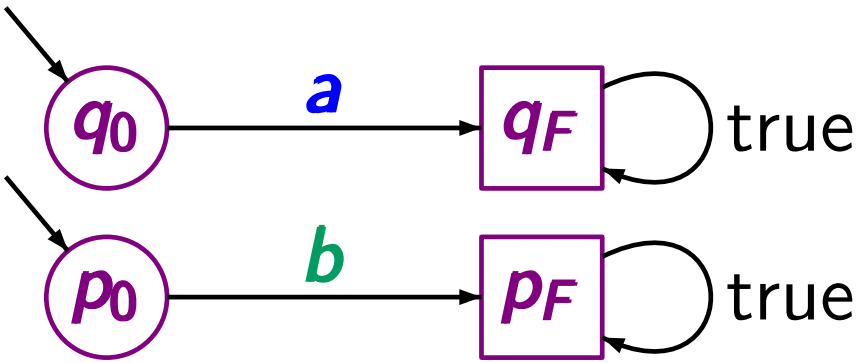


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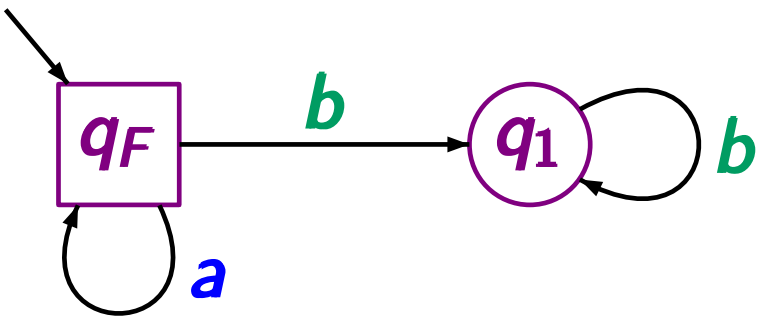


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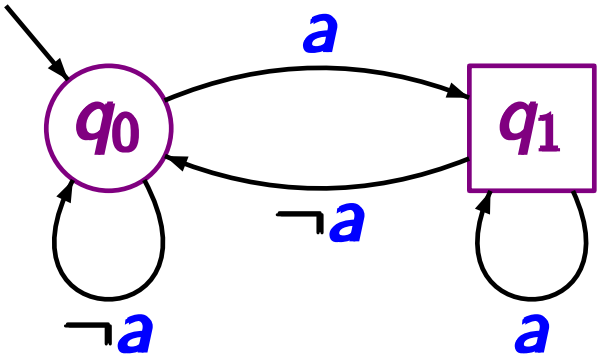


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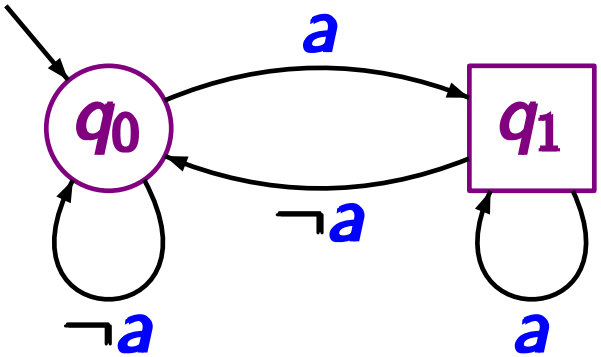


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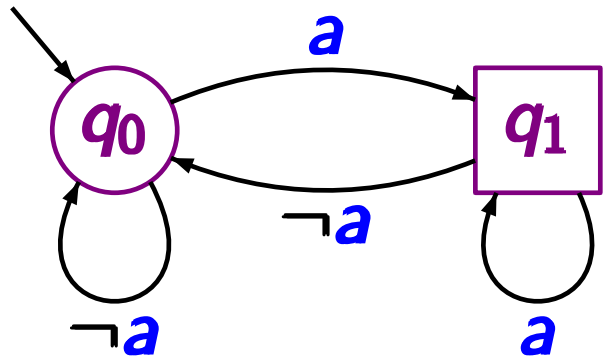


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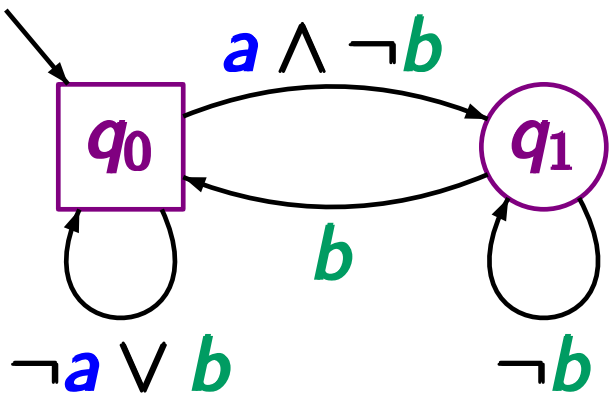


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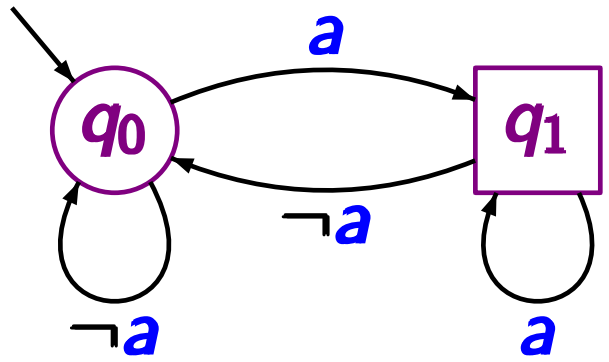


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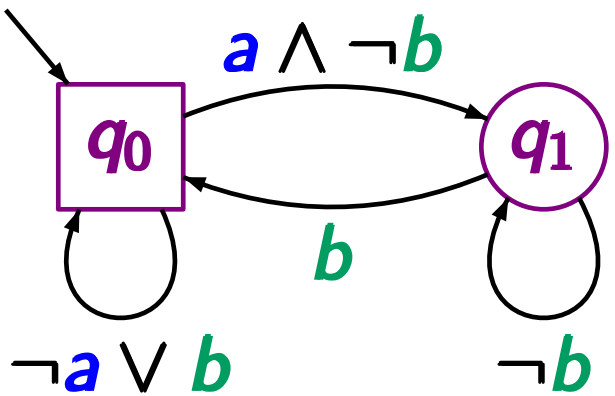


$$\mathcal{L}_\omega(\mathcal{A}) = ?$$

e.g., $\varnothing\,\varnothing\,\varnothing\,\varnothing\ldots = \varnothing^\omega$

$(\{a\}\,\{b\})^\omega$

} are accepted by $\mathcal{A}$

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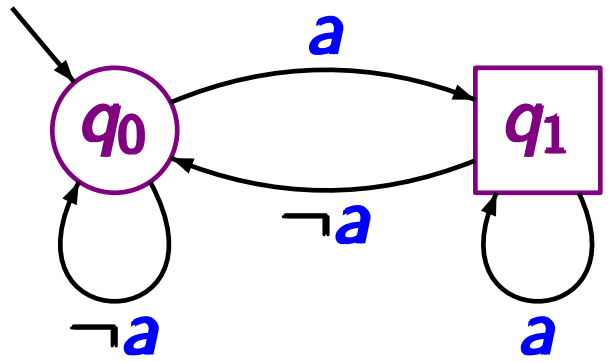


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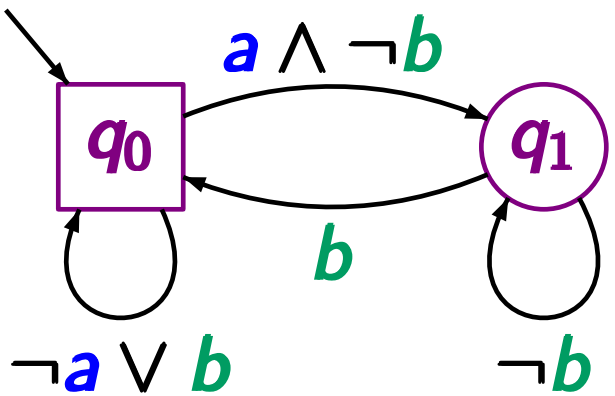



$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\Box(a \rightarrow \Diamond b))$$

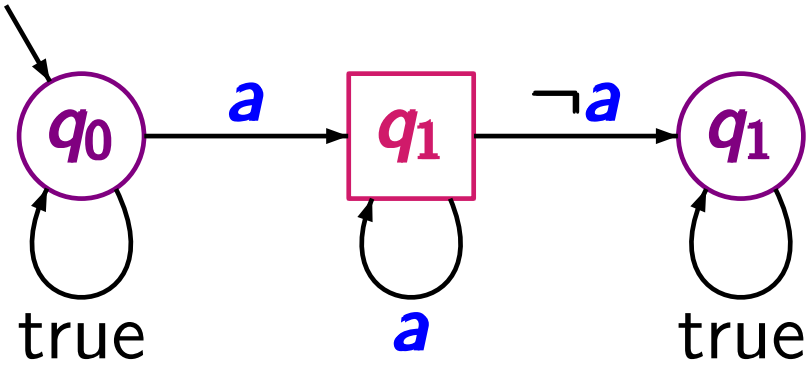
e.g., $\varnothing\,\varnothing\,\varnothing\,\varnothing\ldots = \varnothing^\omega$

$(\{a\}\,\{b\})^\omega$

} are accepted by $\mathcal{A}$

# NBA for LTL formula

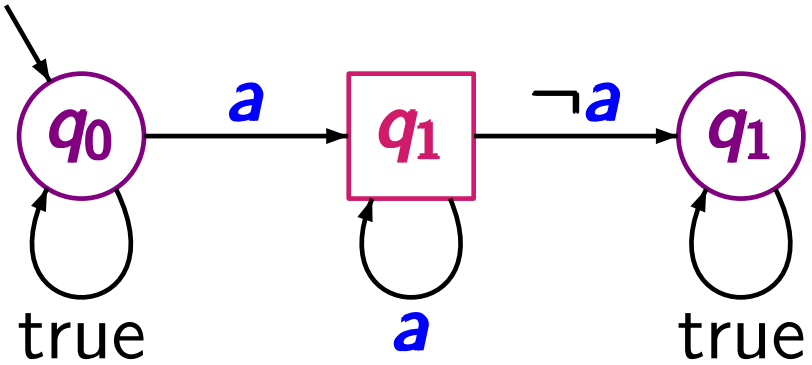





$$\mathcal{L}_\omega(\mathcal{A}) = \,?$$

# NBA for LTL formula



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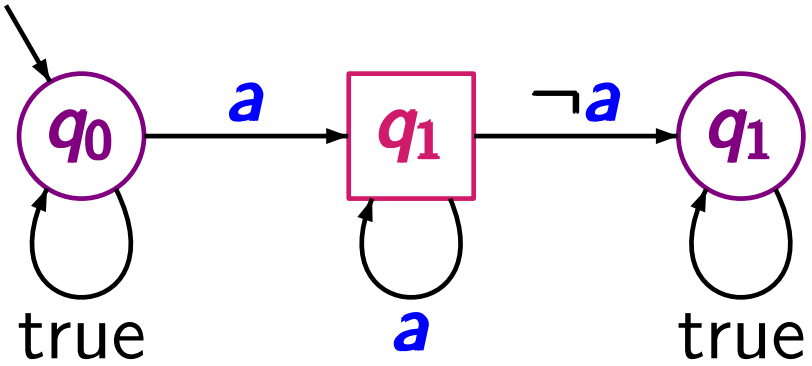


$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\Diamond\Box a)$$

possible runs for $\{a\}^\omega$

| | |
|---|---|
| $q_0\ q_0\ q_0\ q_0\ q_0\ q_0\ \ldots$ | not accepting |
| $q_0\ q_1\ q_1\ q_1\ q_1\ q_1\ \ldots$ | accepting |
| $q_0\ q_0\ q_1\ q_1\ q_1\ q_1\ \ldots$ | accepting |
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| $\vdots$ | |

# NFA and NBA for safety properties

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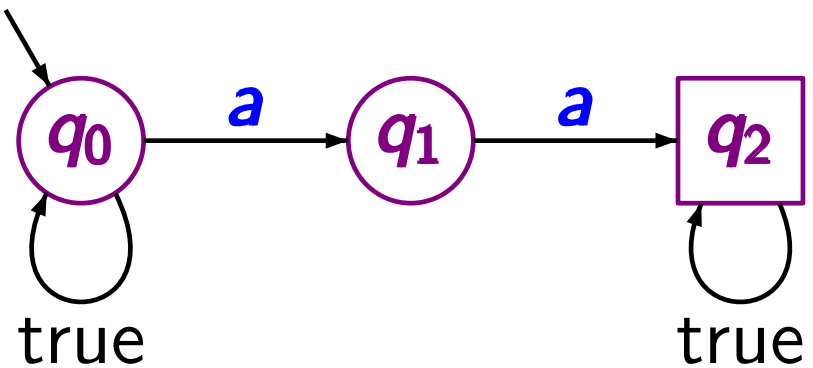

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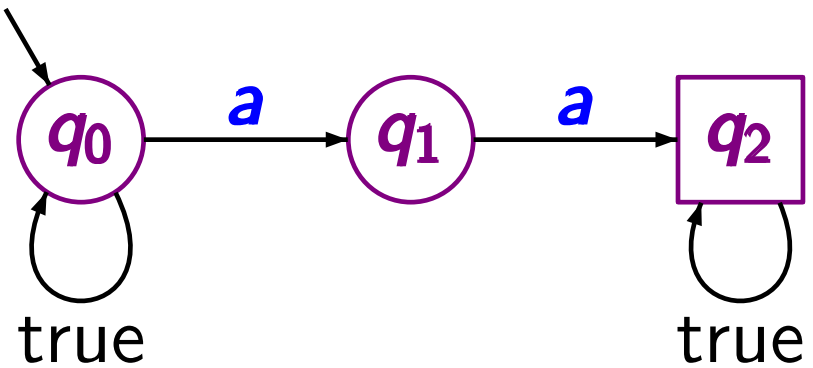


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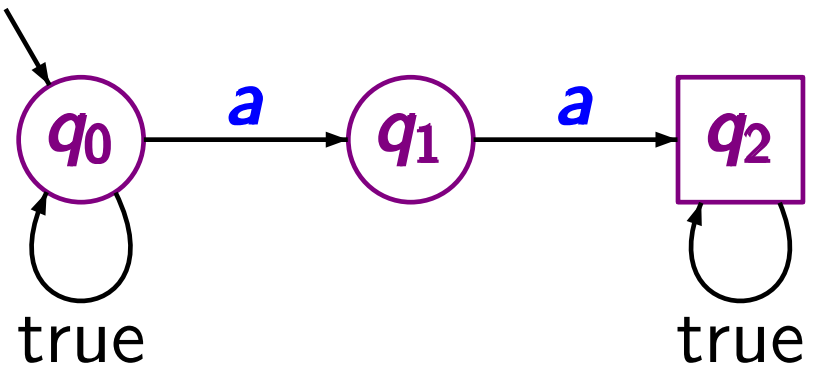


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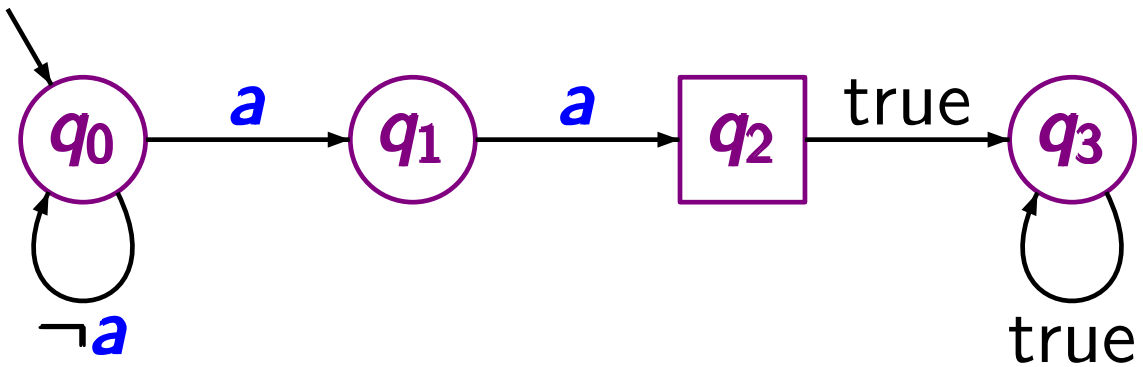


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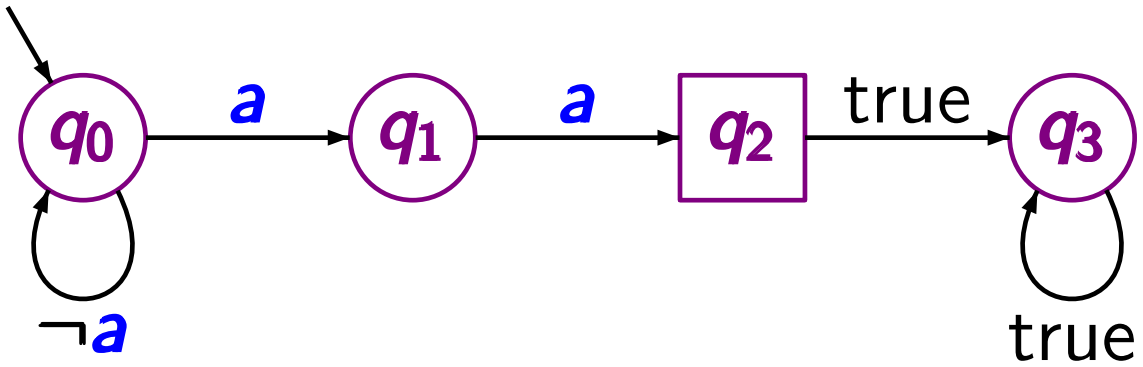


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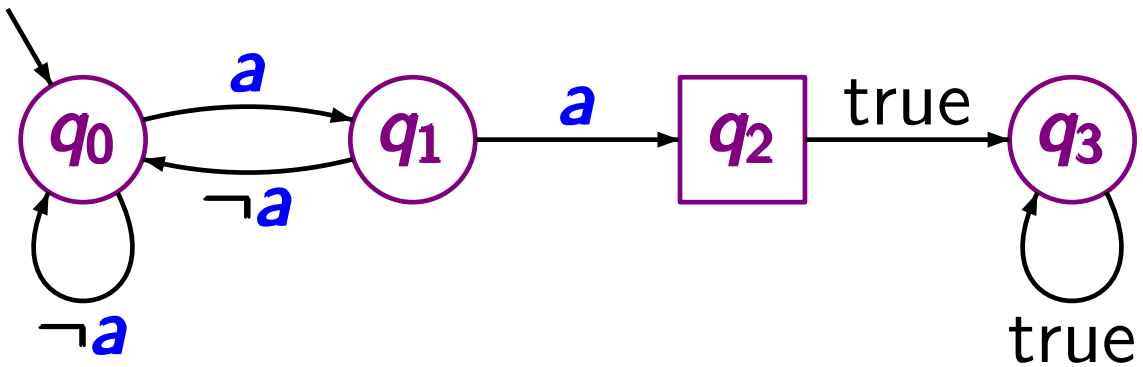


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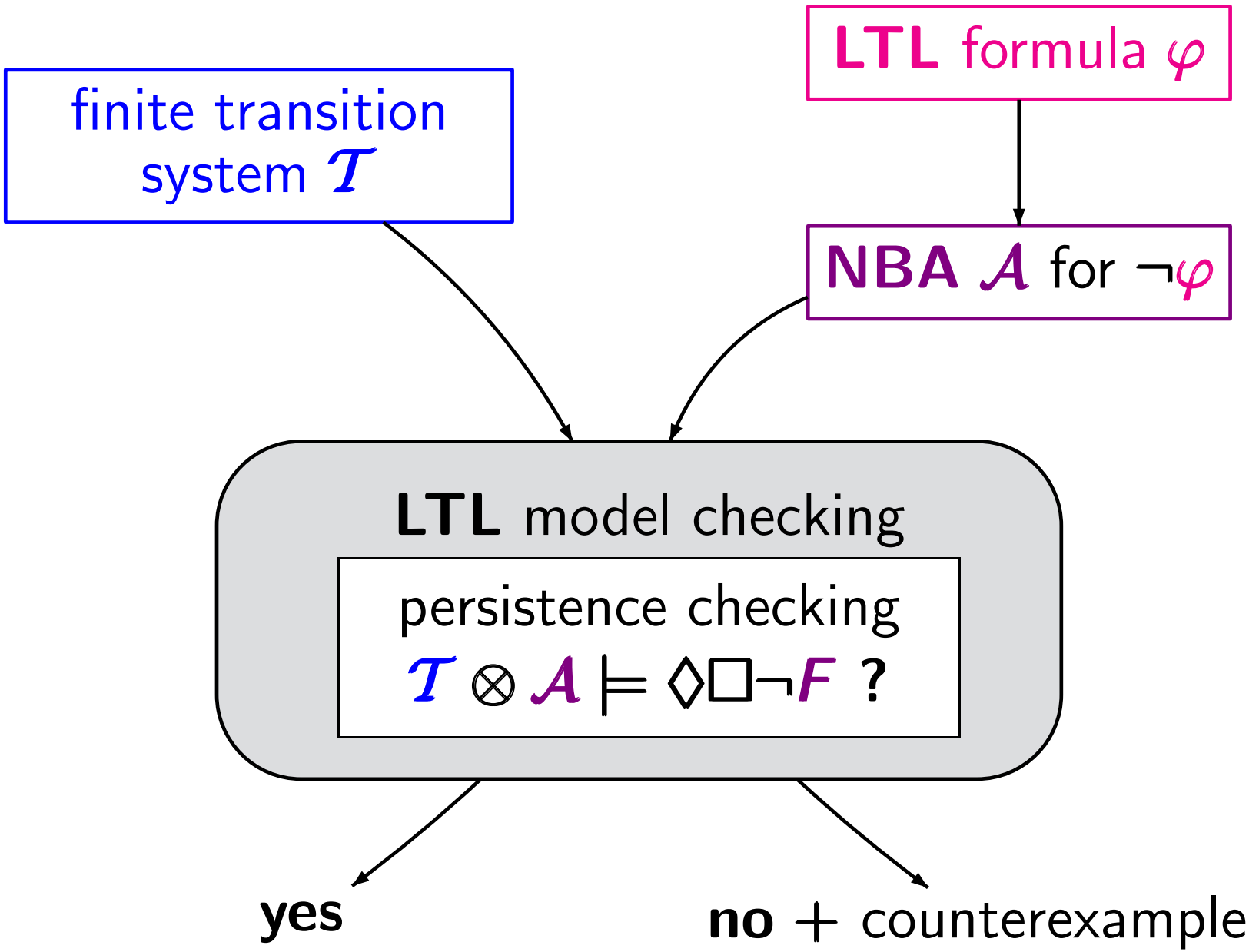
# LTL model checking





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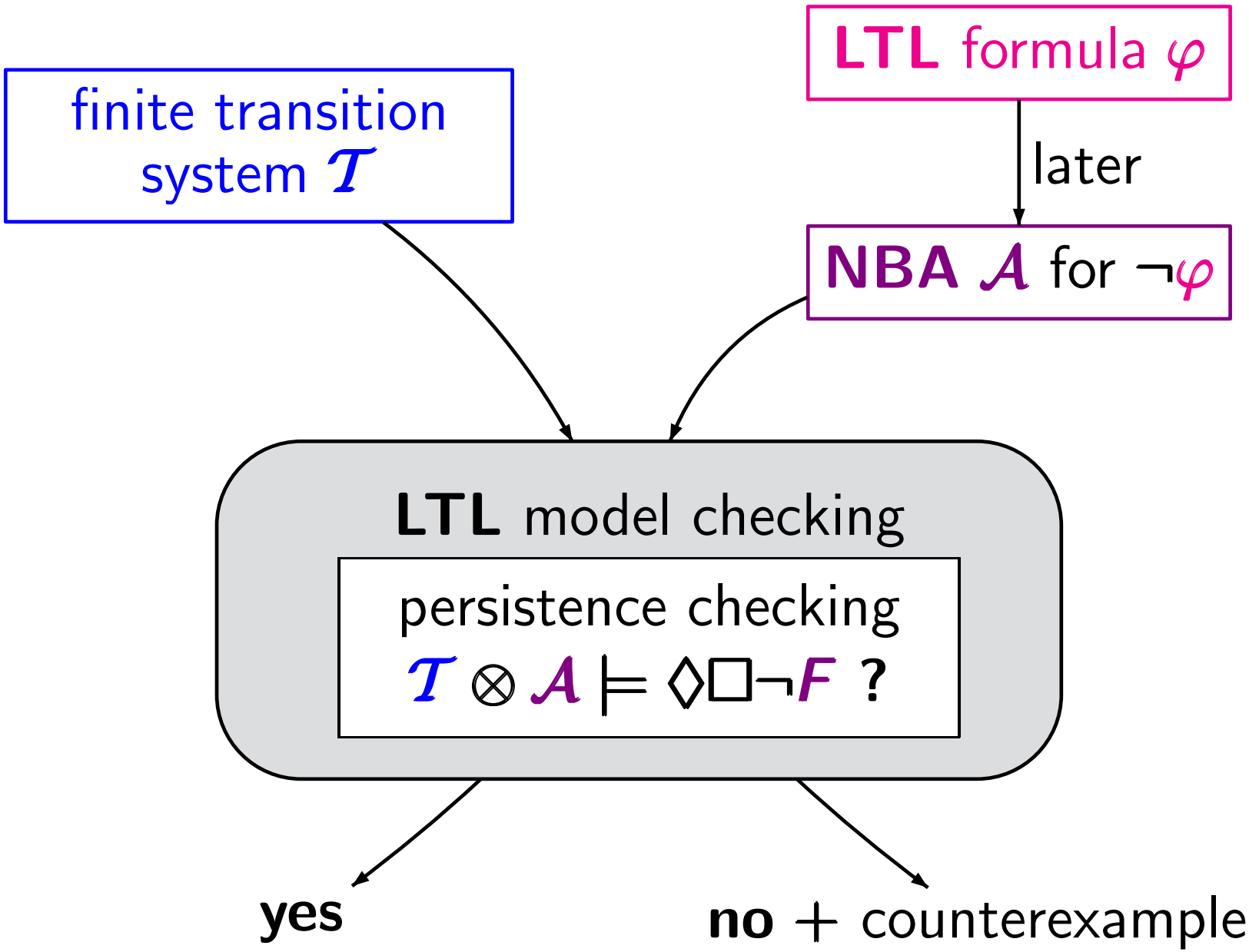

# Recall: product transition system



$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

$\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, F)$ NBA or NFA

non-blocking, $Q_0 \cap F = \varnothing$

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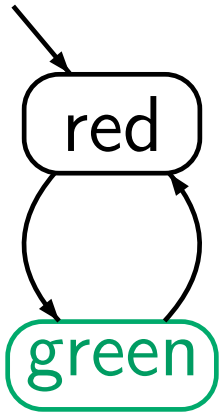
transition relation:

$$\frac{s \xrightarrow{\alpha} s' \ \wedge \ q' \in \delta(q, L(s'))}{\langle s, q\rangle \xrightarrow{\alpha}' \langle s', q'\rangle}$$

# Example: LTL model checking



TS $\mathcal{T}$




**LTL** formula $\varphi = \Box\Diamond$green

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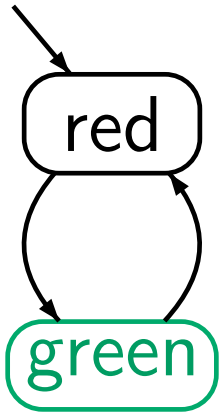


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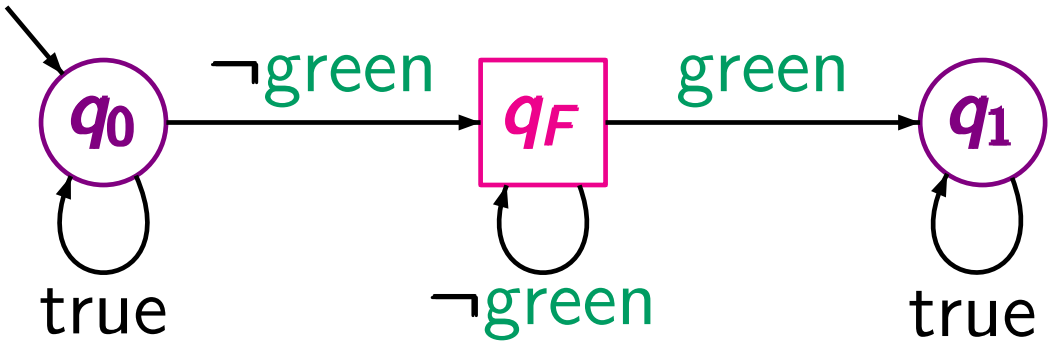

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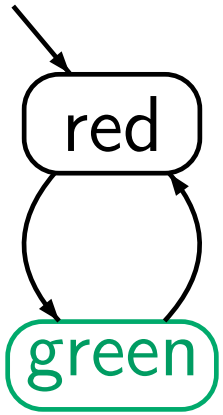


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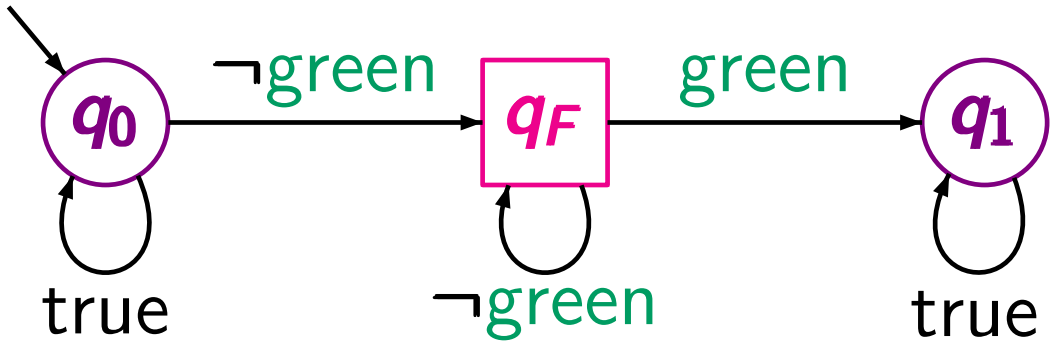


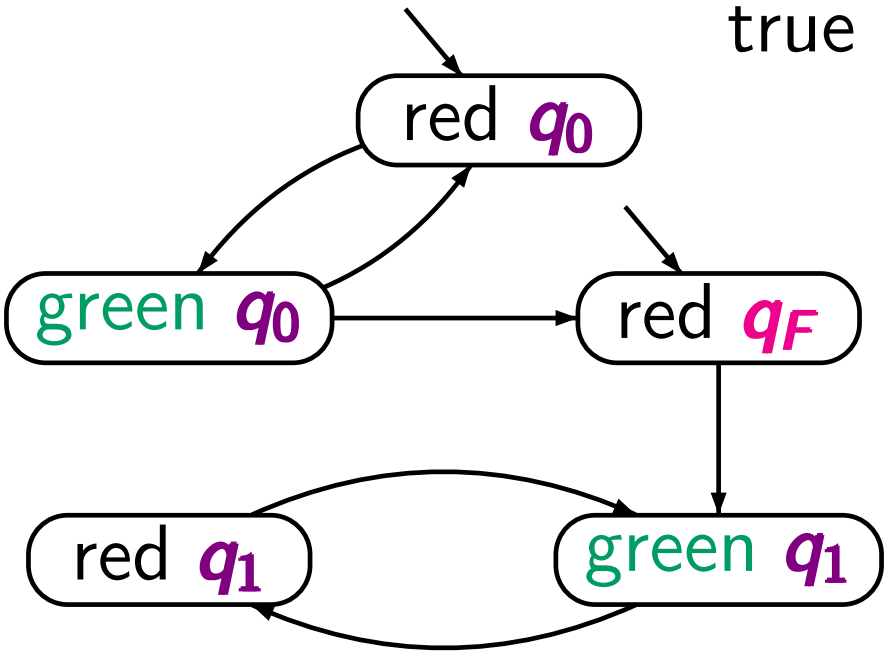


reachable fragment of the product TS $\mathcal{T} \otimes \mathcal{A}$

# Example: LTL model checking



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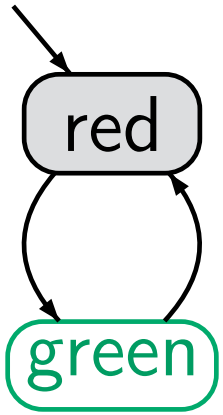


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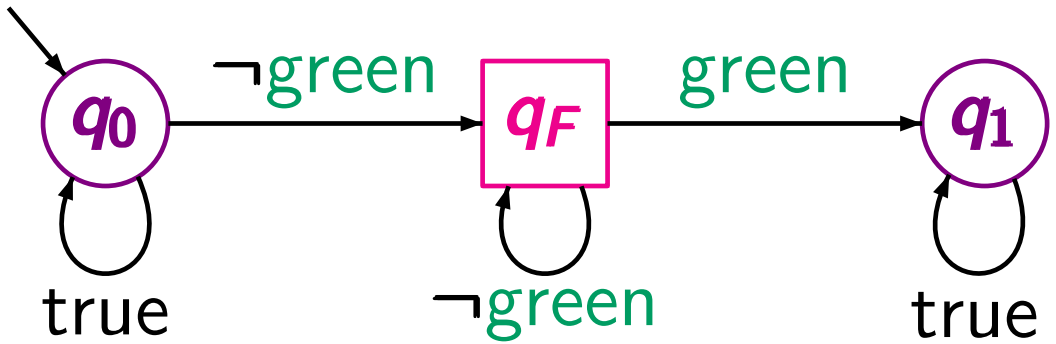


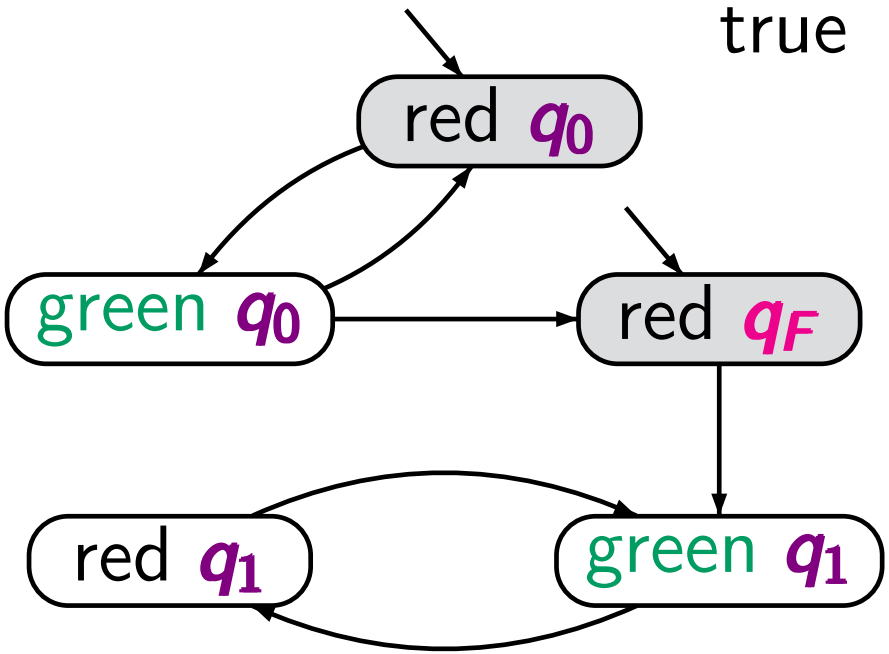


initial states:

$\langle \text{red}, q \rangle$ where

$$q \in \delta(q_0, L(\text{red})) = \delta(q_0, \varnothing) = \{q_0, q_F\}$$

# Example: LTL model checking



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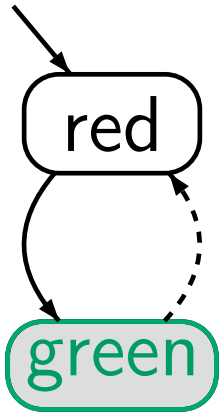


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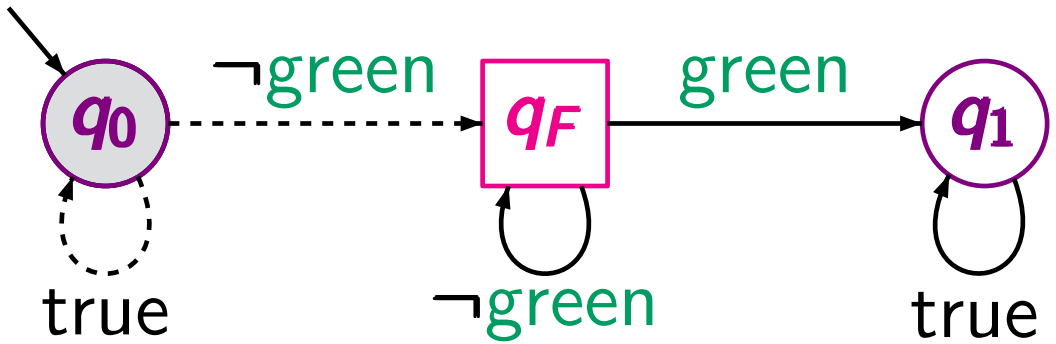


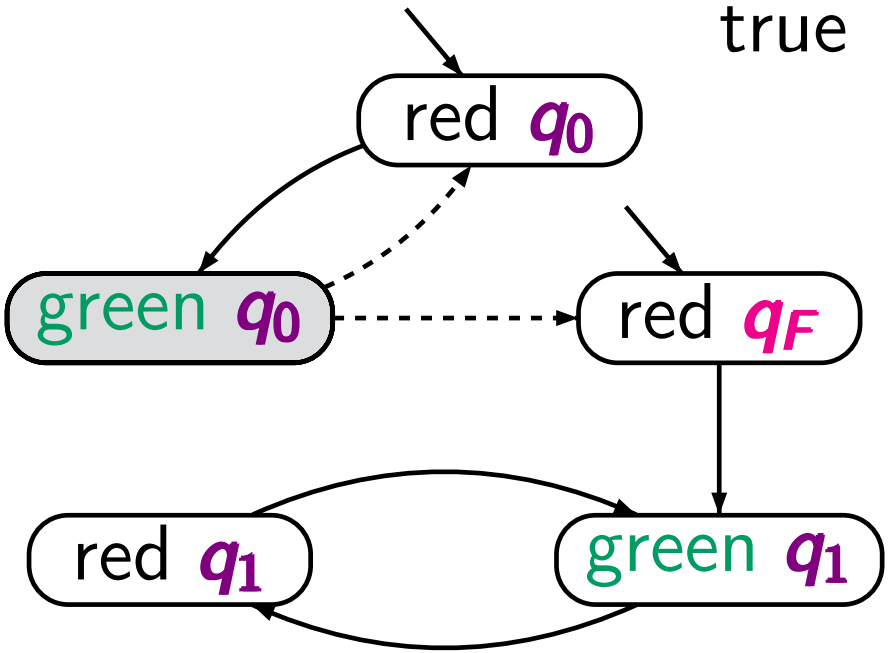


transition

$\langle \text{green}, q_0 \rangle \rightarrow \langle \text{red}, q \rangle$

$q \in \delta(q_0, L(\text{red}))$

$= \delta(q_0, \varnothing)$

$= \{q_0, q_F\}$

# Example: LTL model checking



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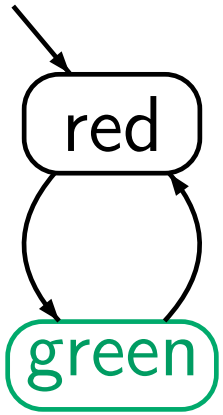


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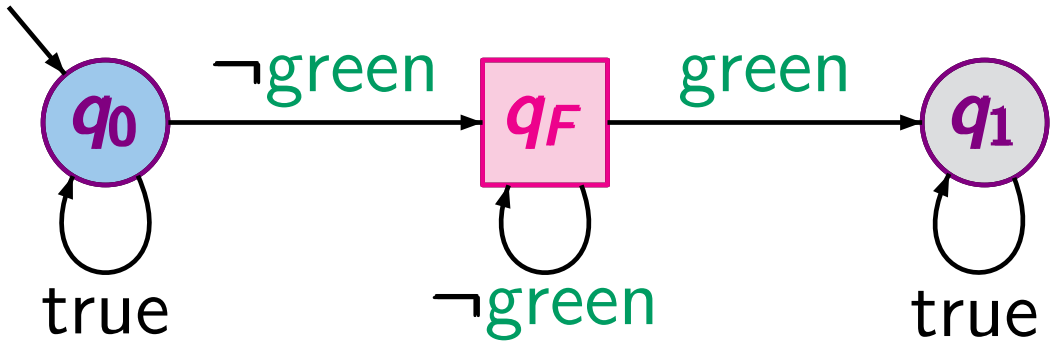


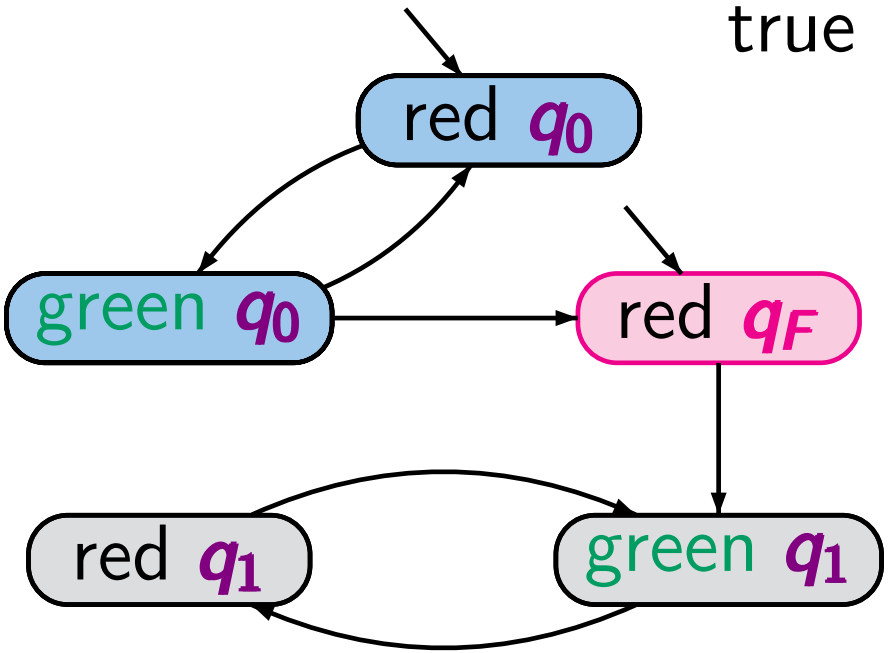


atomic propositions

$AP' = \{q_0, q_F, q_1\}$

obvious labeling function

# Example: LTL model checking



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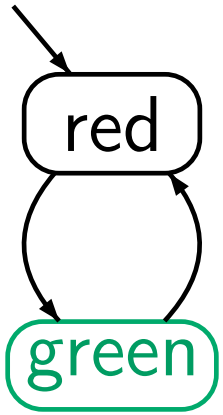


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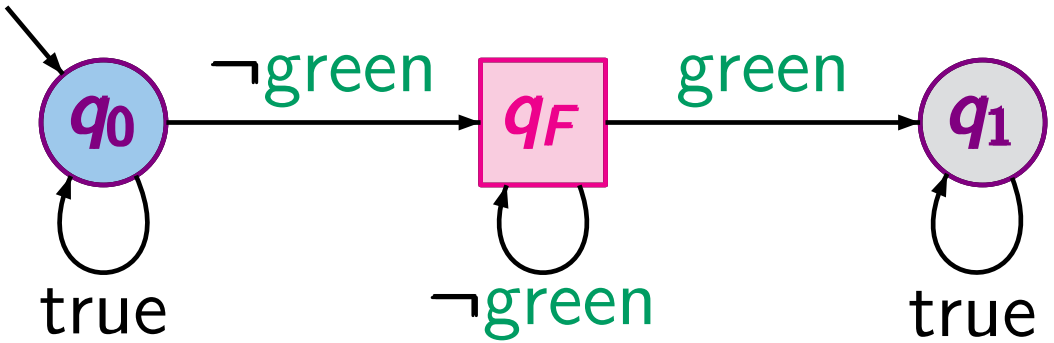


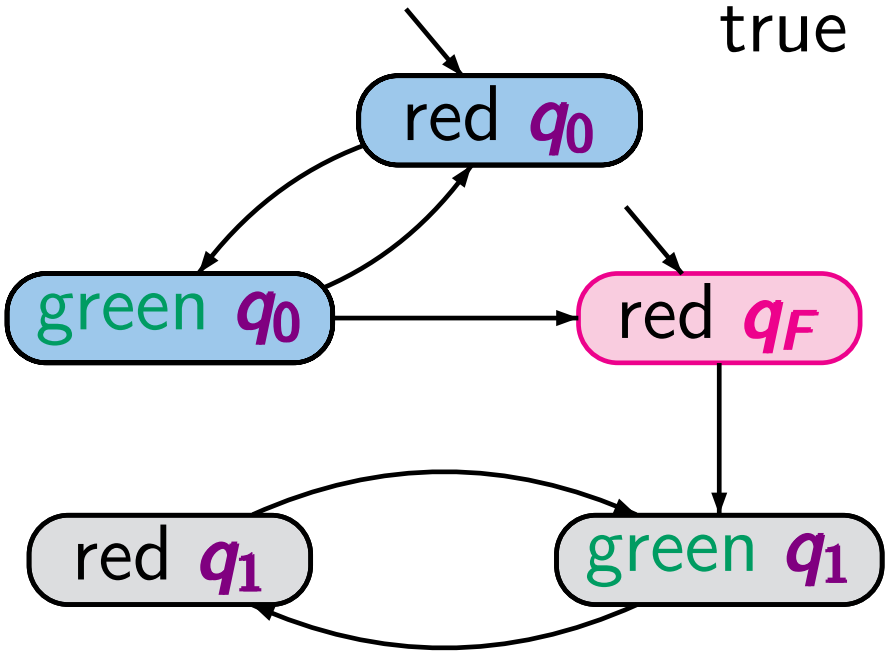


$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

# Example: LTL model checking



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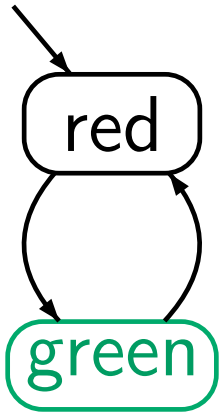


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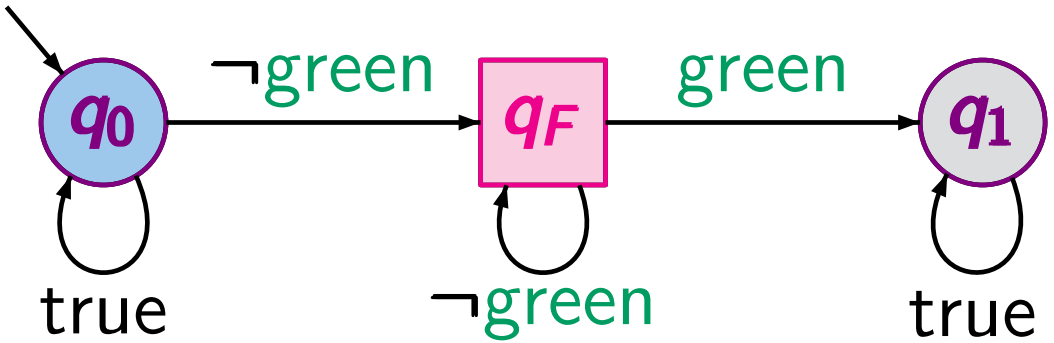
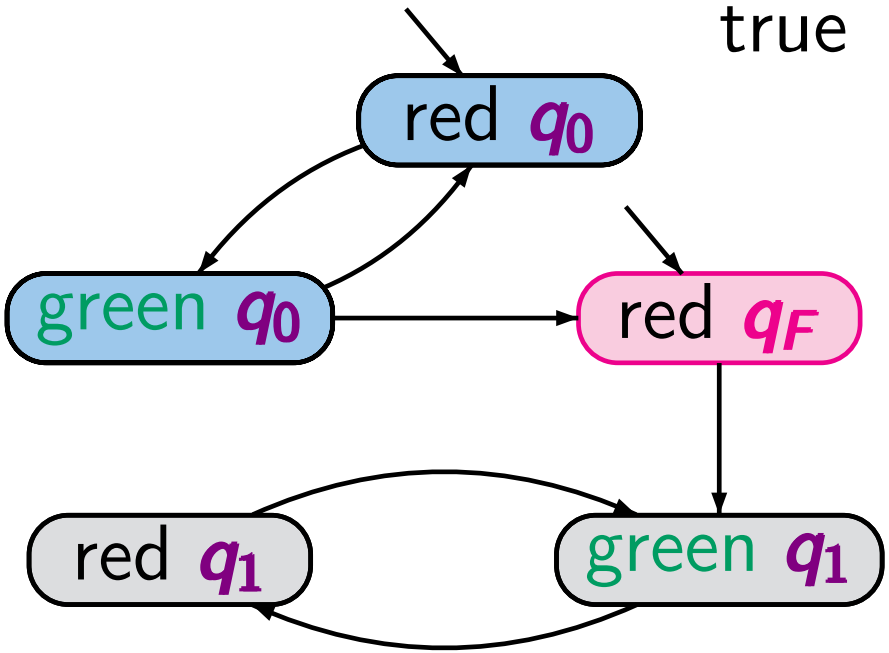


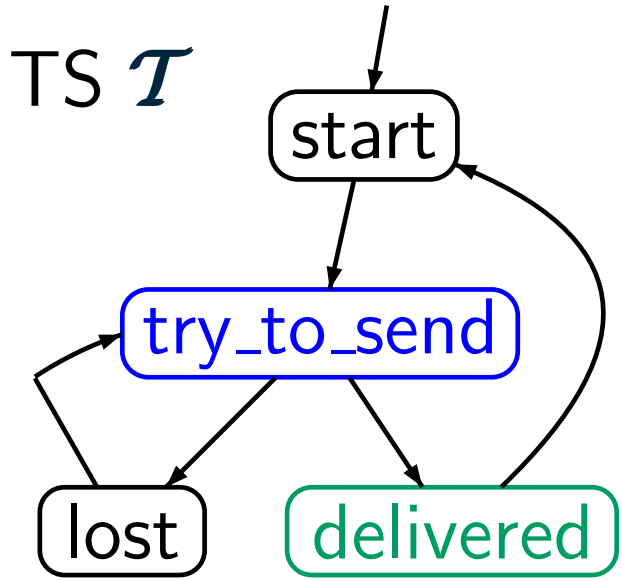




$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

hence: $\mathcal{T} \models \varphi$

# Example: LTL model checking



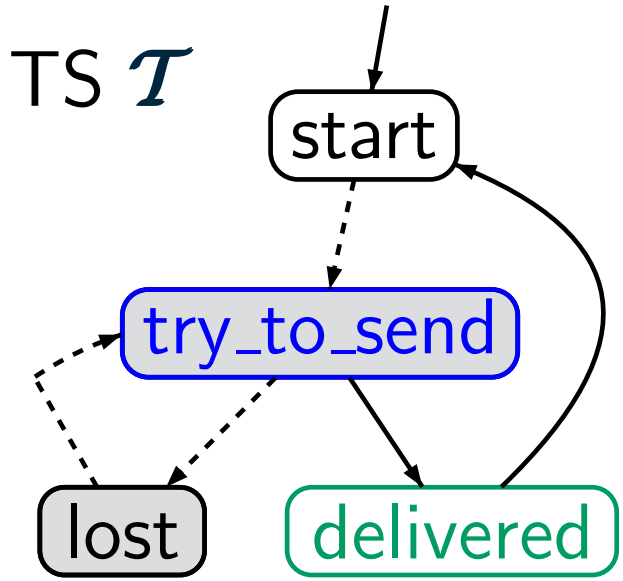



**LTL** formula $\varphi = \Box(\mathit{try} \rightarrow \Diamond \mathit{del})$

"each (repeatedly) sent message will eventually be delivered"

# Example: LTL model checking



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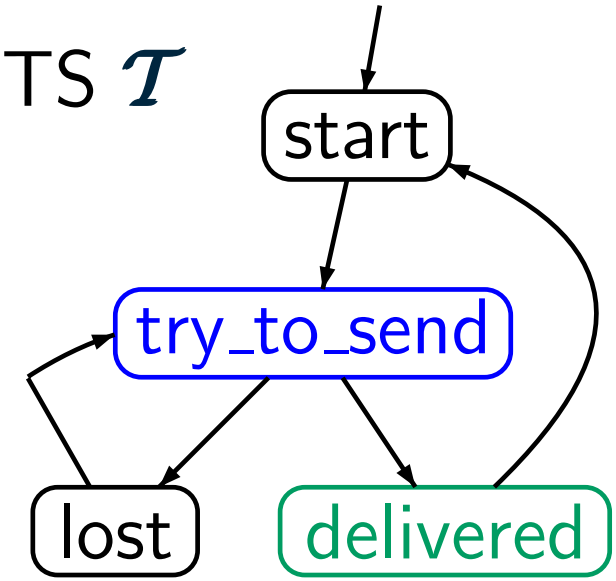
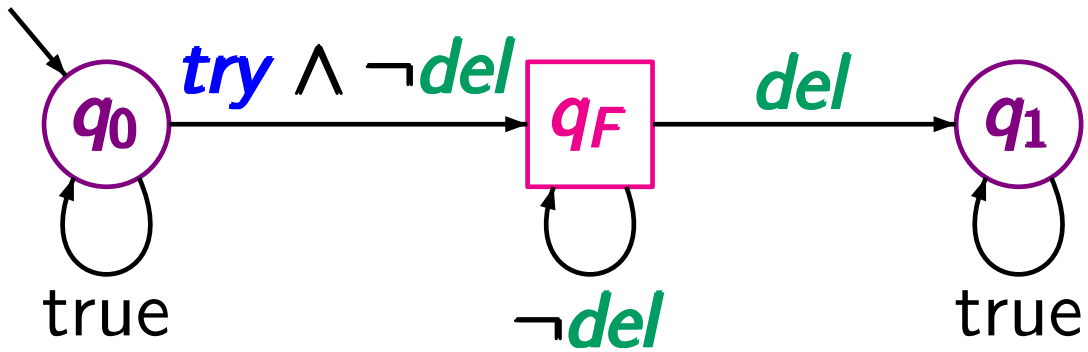



NBA $\mathcal{A}$ for $\neg\varphi \equiv \Diamond(\mathit{try} \wedge \Box\neg\mathit{del})$

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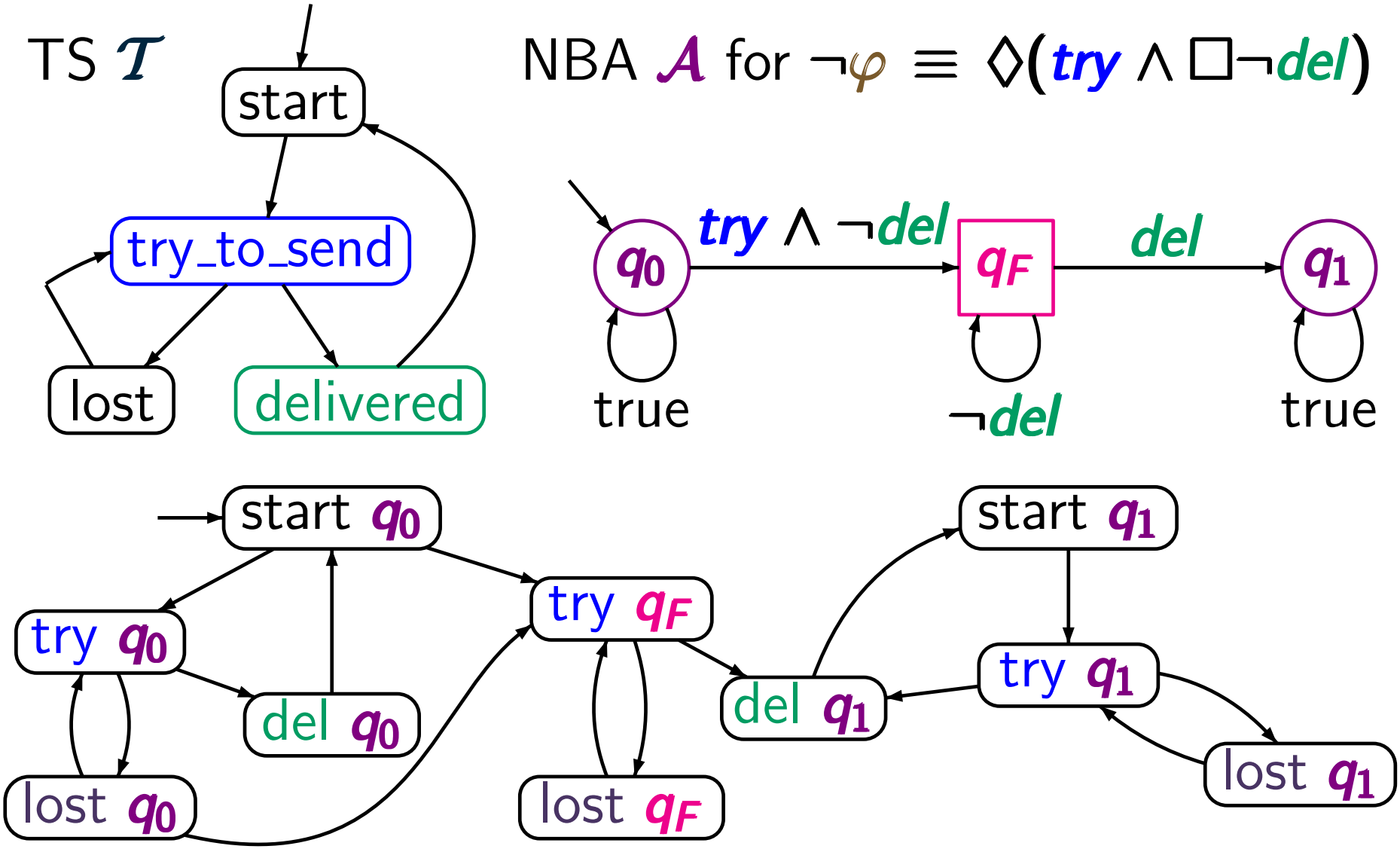
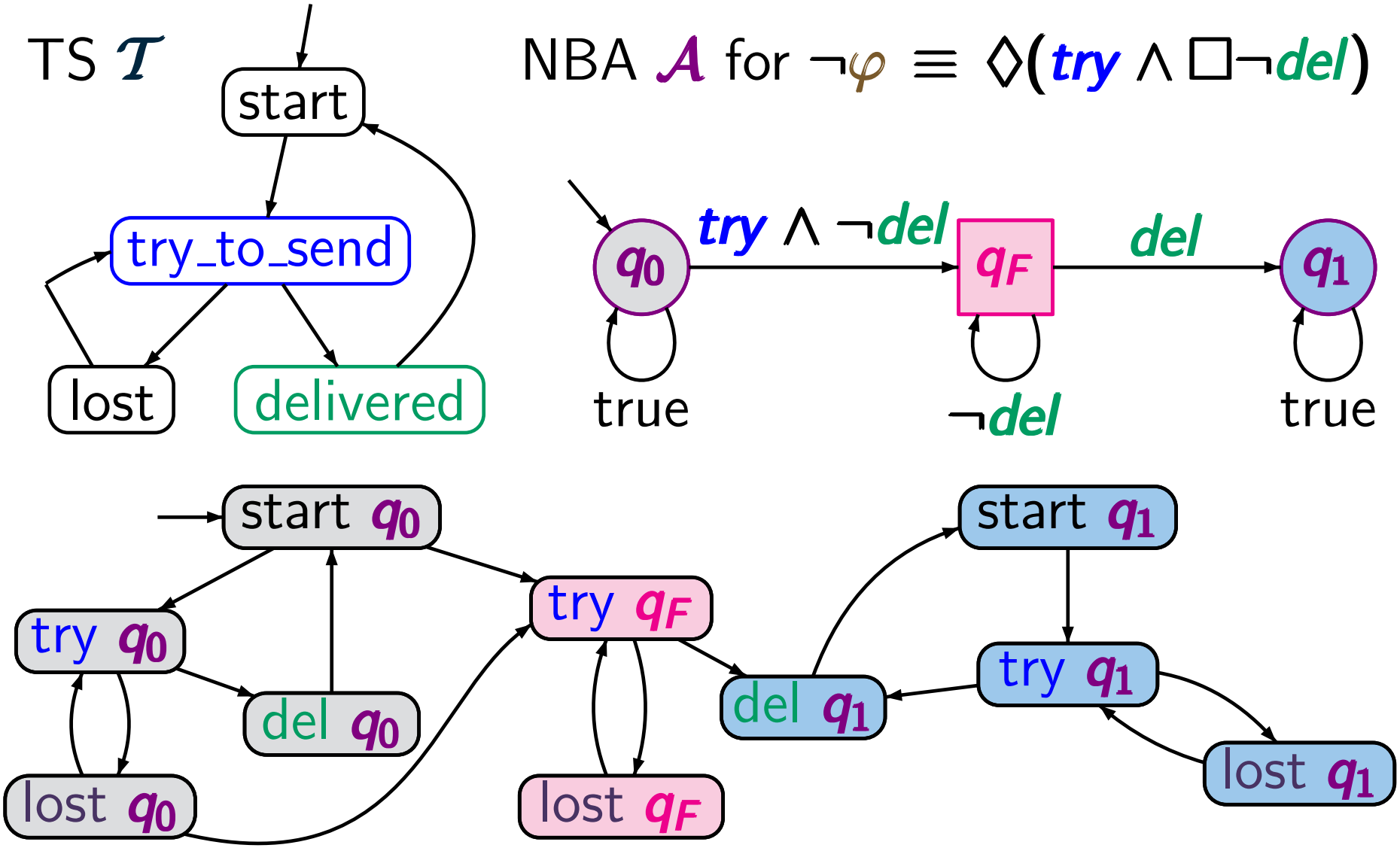



reachable fragment of the product-TS

# Example: LTL model checking

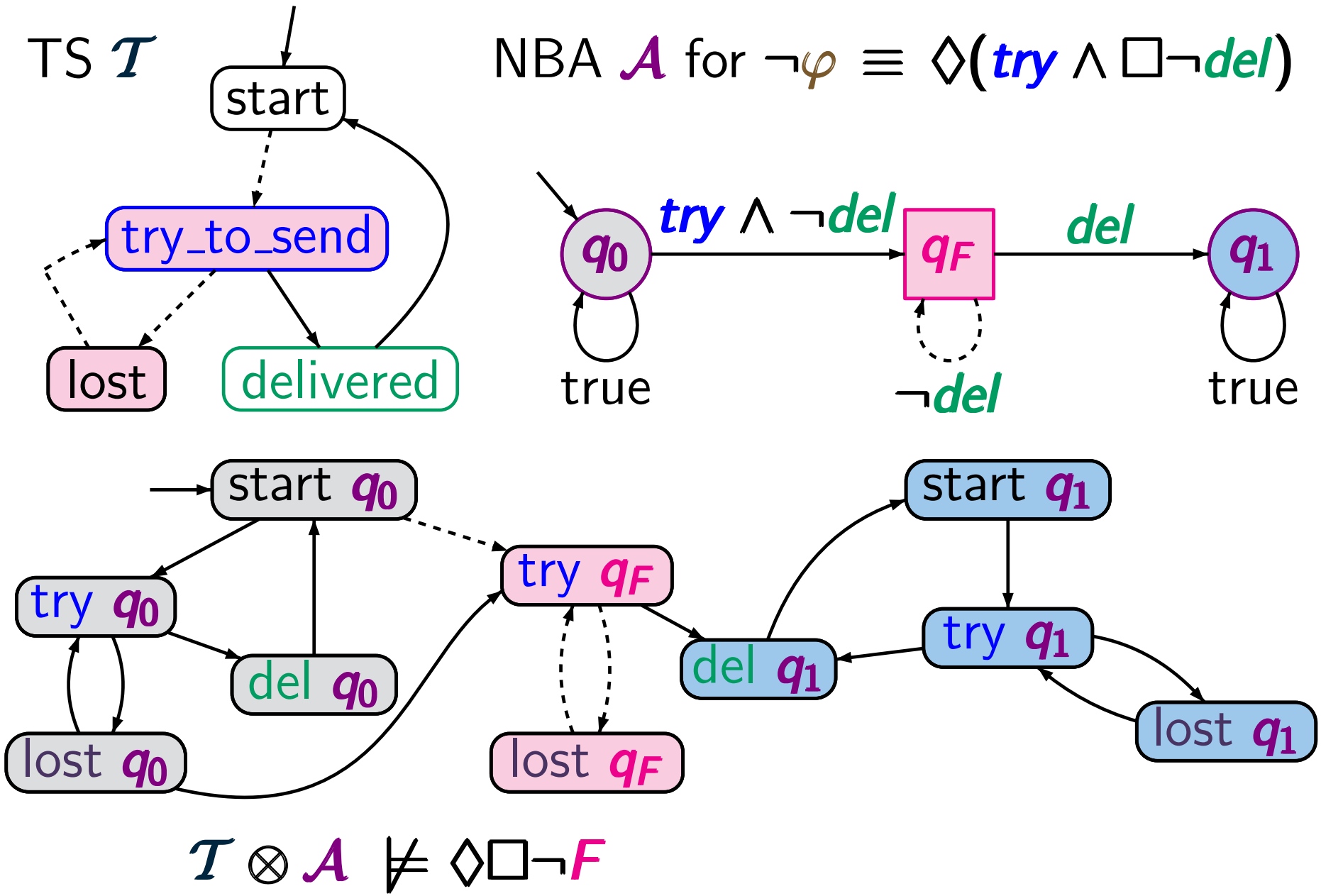





set of atomic propositions $AP' = \{q_0, q_1, q_F\}$

# Example: LTL model checking





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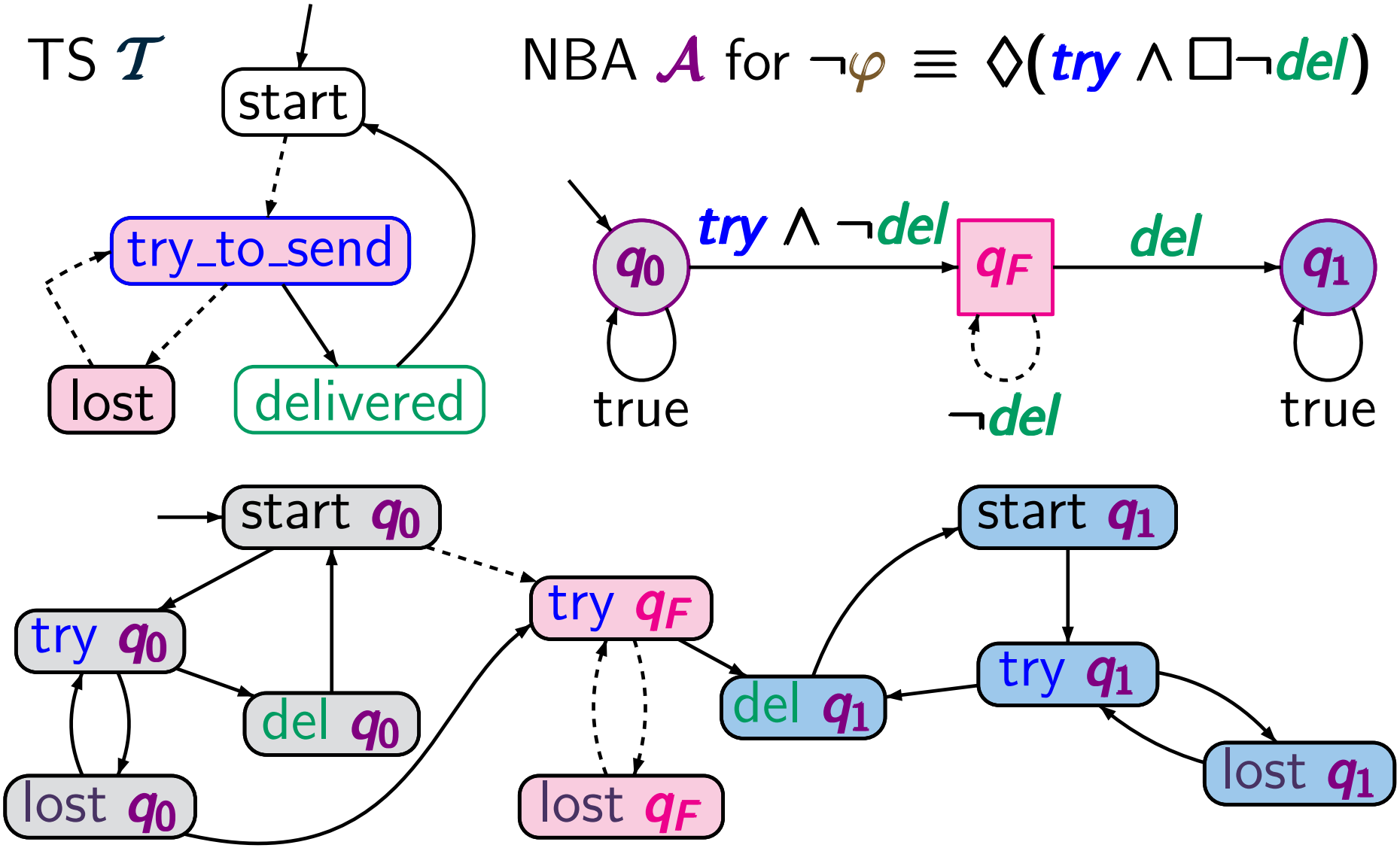


$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$ hence: $\mathcal{T} \not\models \varphi$

# Complexity of LTL model checking



main steps of automata-based LTL model checking:

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**LTL** model checking problem

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we show

- just for fun: **coNP**-hardness
- **PSPACE**-completeness

# Recall: complexity classes

# Complexity classes *P*, *NP*

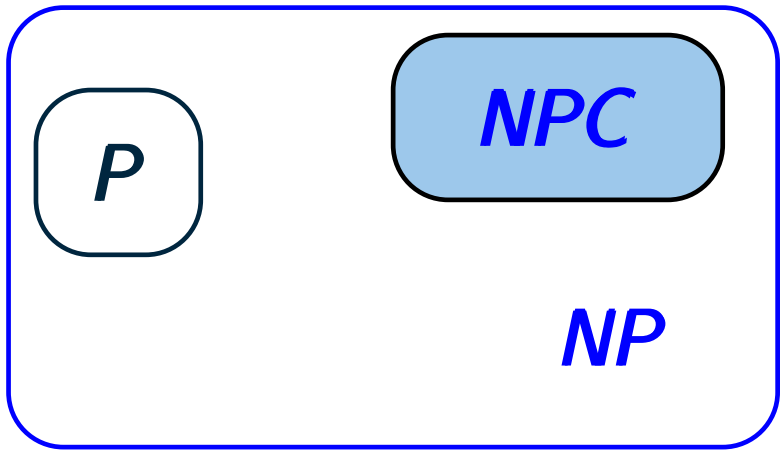



***P*** = class of decision problem solvable in deterministic polynomial time

***NP*** = class of decision problem solvable in nondeterministic polynomial time

# Complexity classes *P*, *NP*






***NPC*** = class of ***NP***-complete problems

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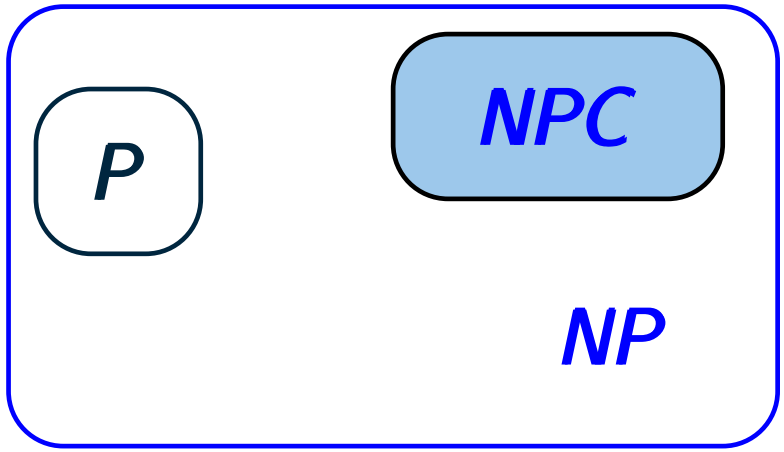


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(1) $L \in NP$

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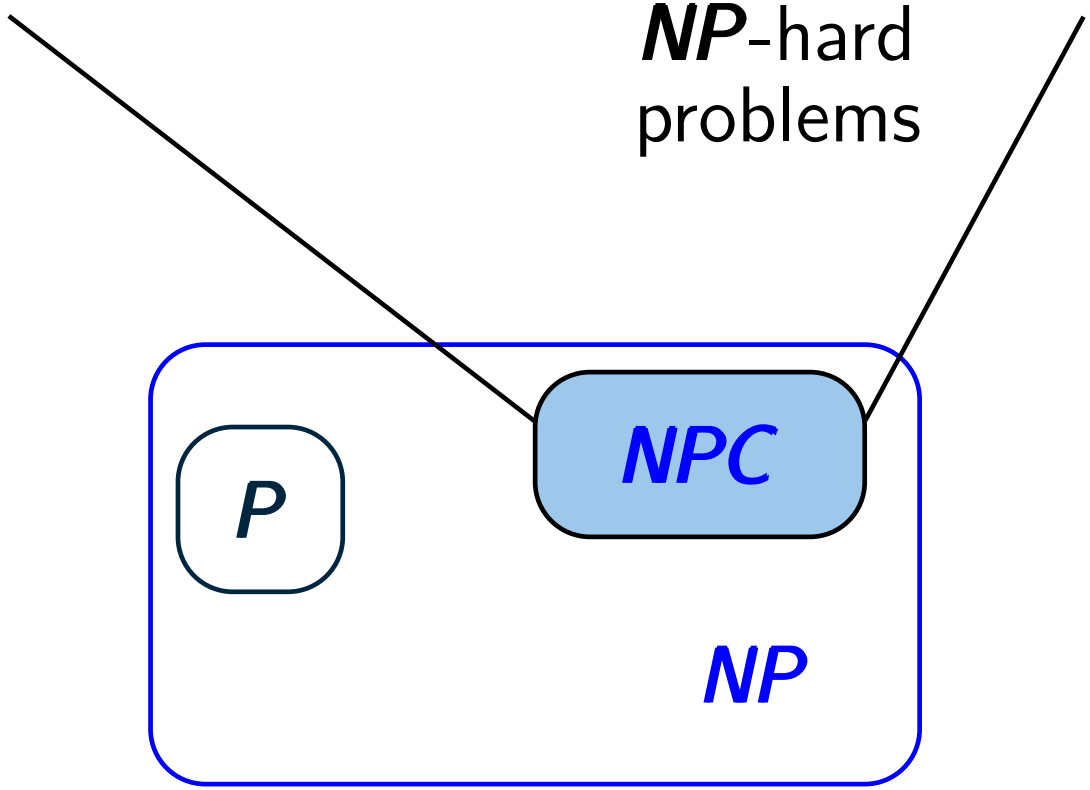


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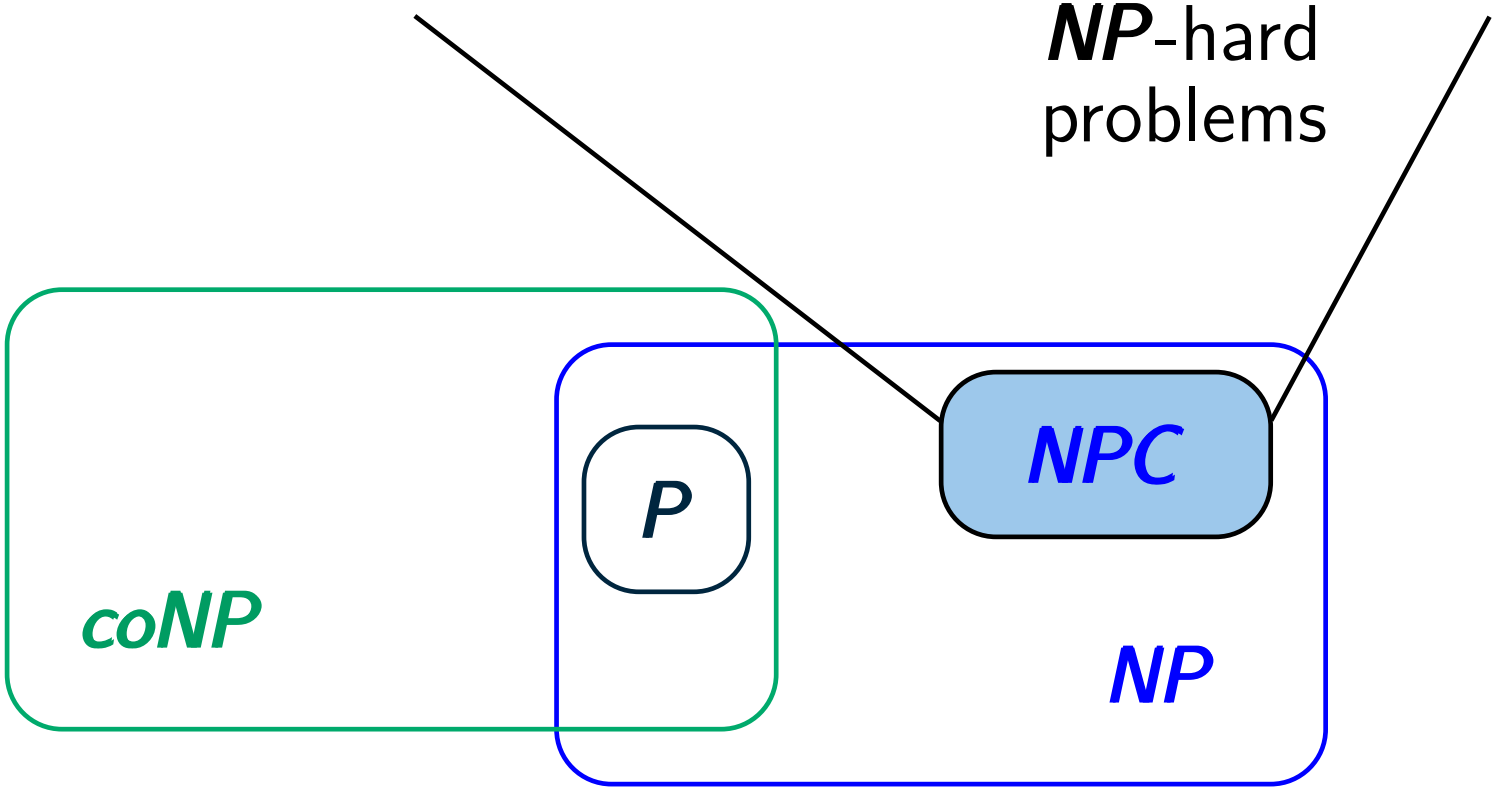
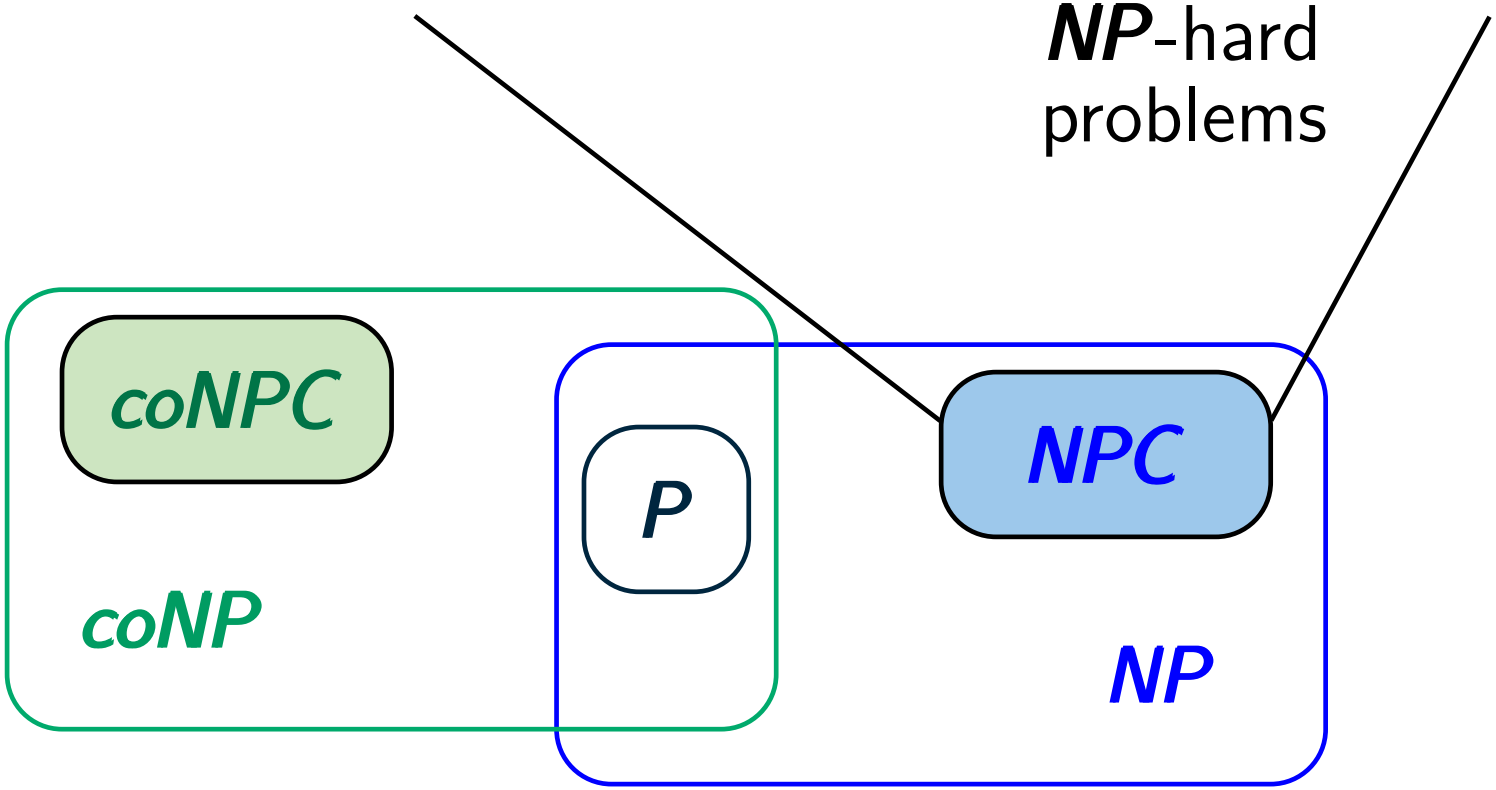



$$coNP = \{ \overline{L} : L \in NP \}$$

$\overline{L}$: complement of $L$

# Complexity classes *P*, *NP*, *coNP*



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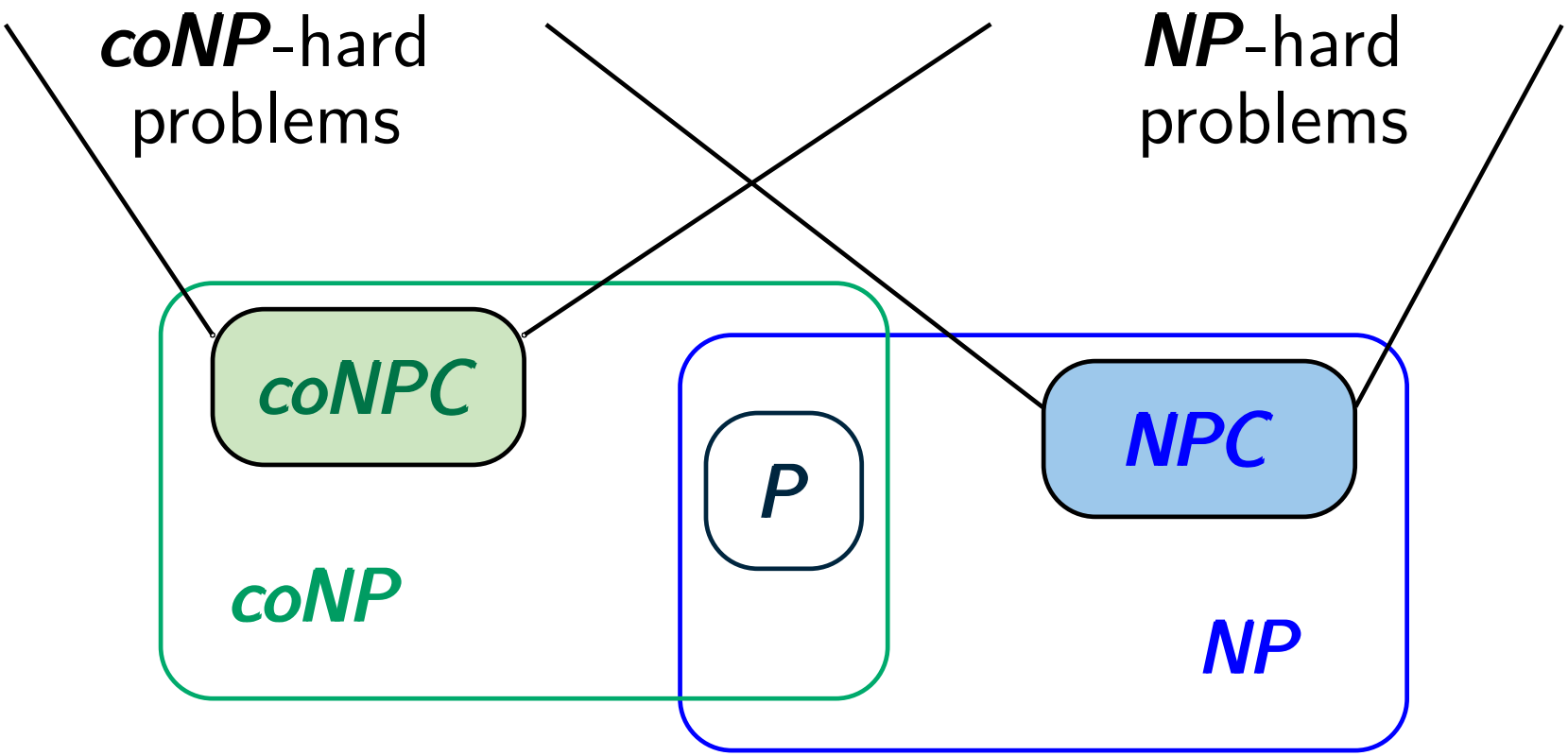


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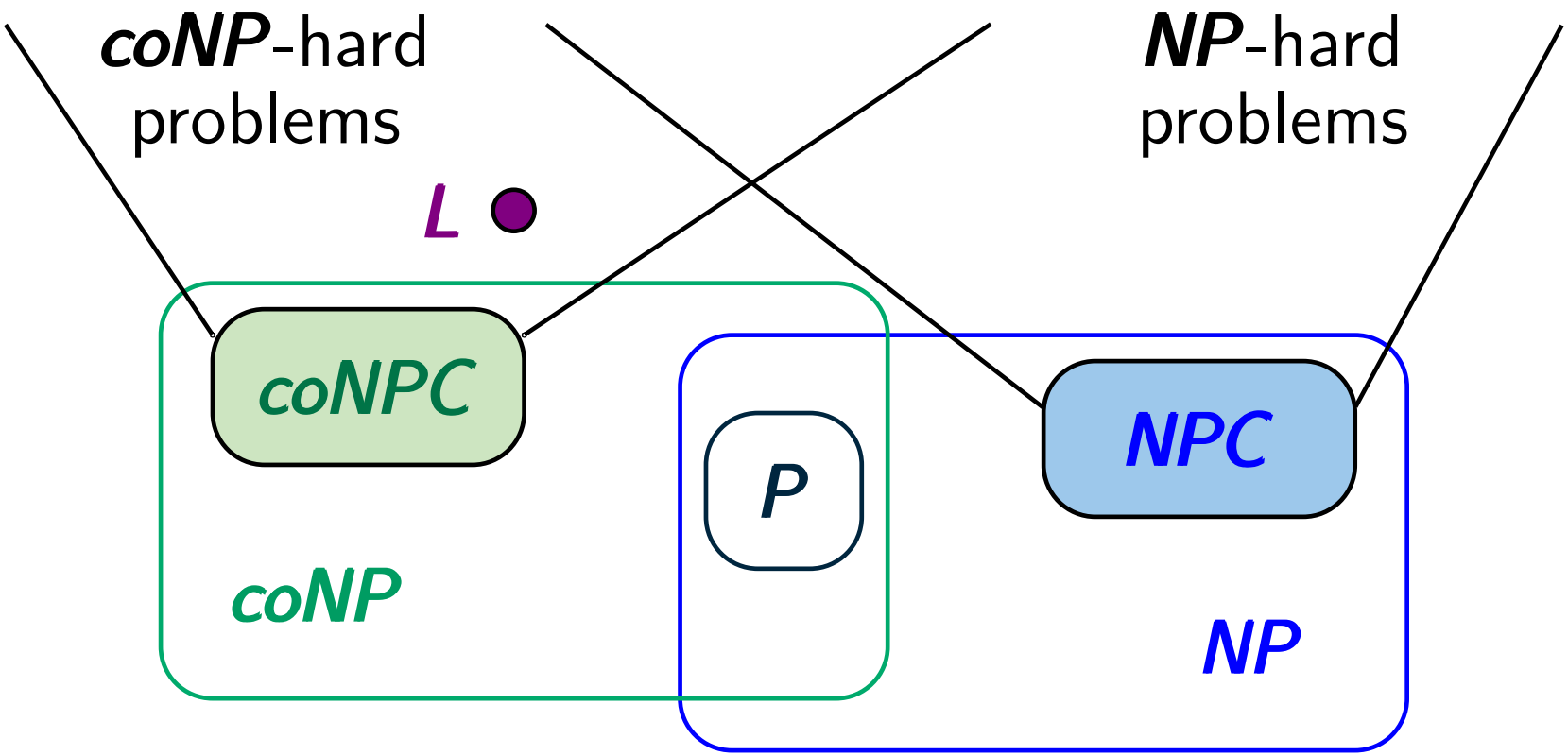


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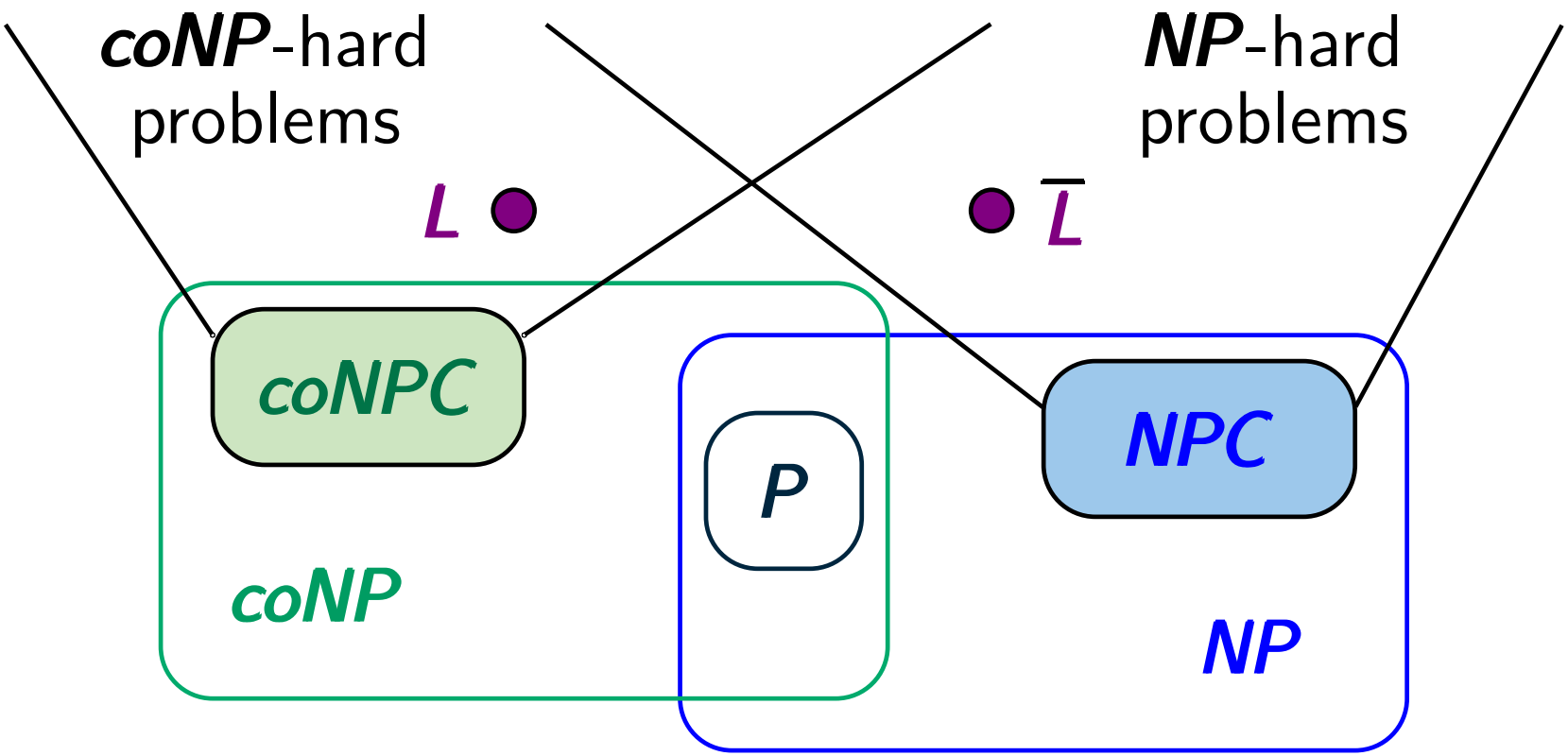


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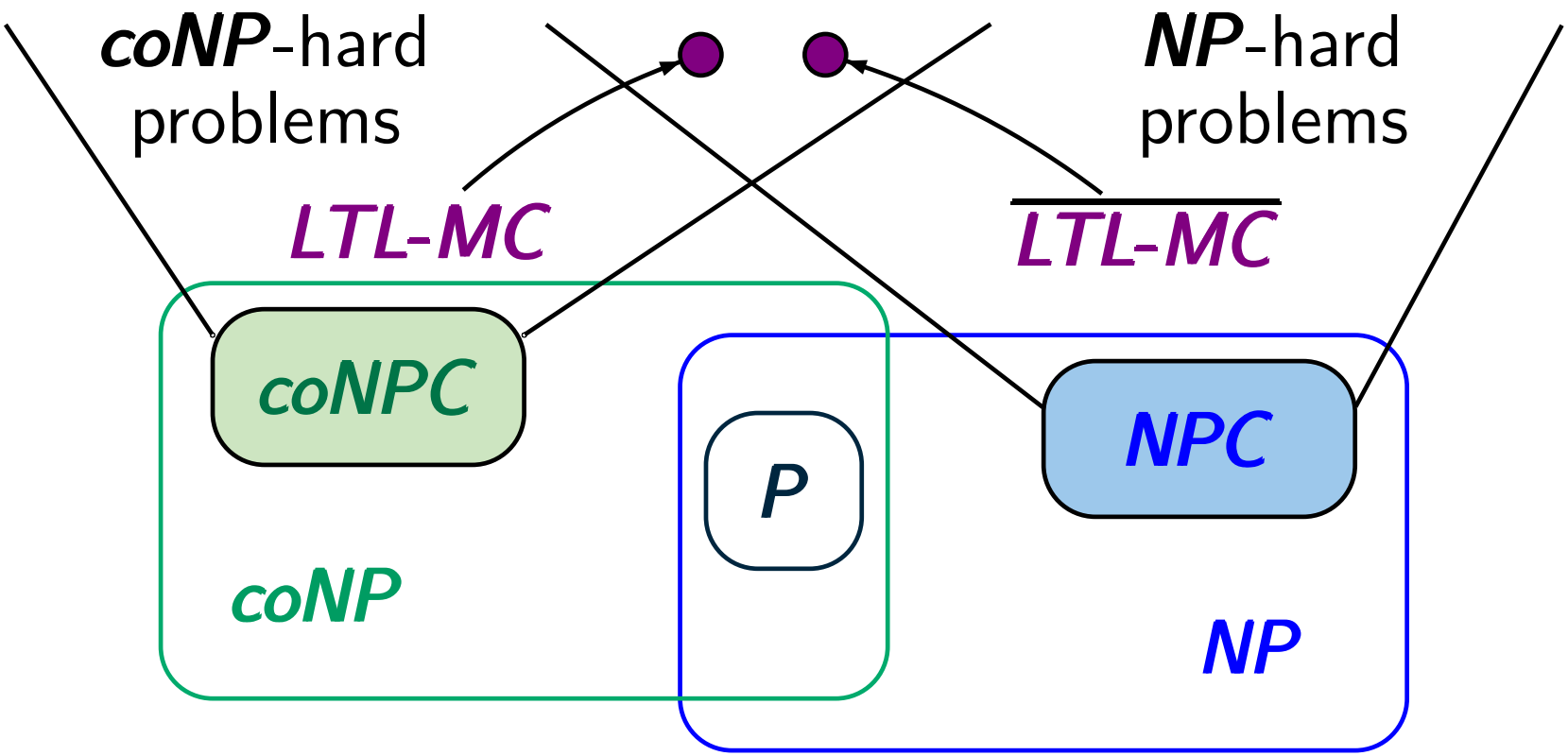


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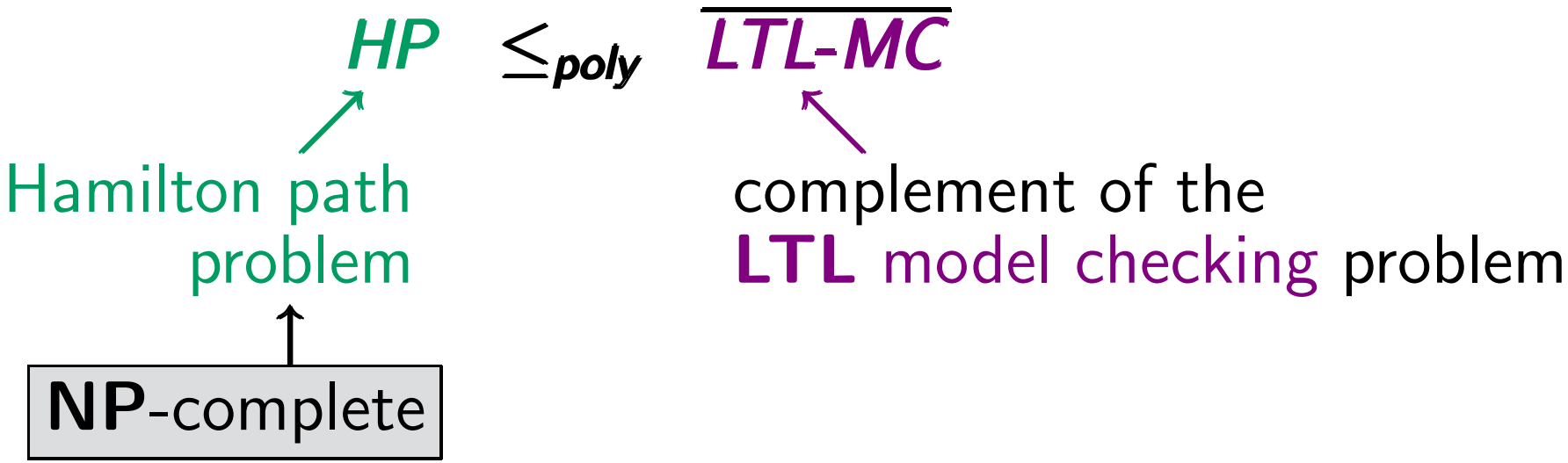


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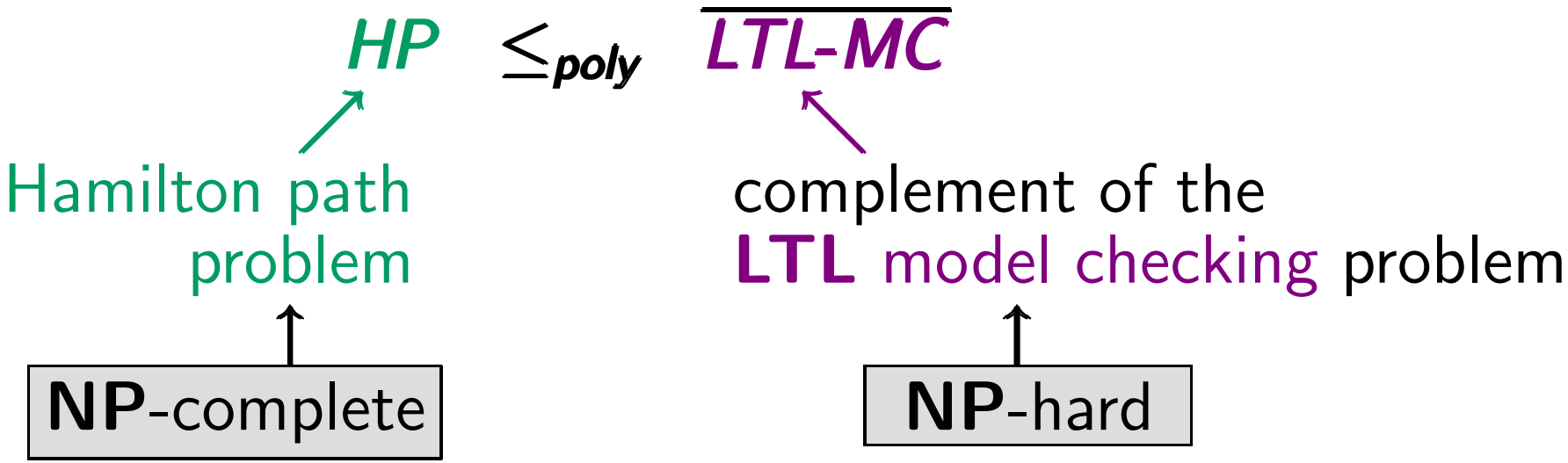


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# Complexity of LTL model checking



We just saw:

> The **LTL** model checking problem is ***coNP***-hard

We now prove:

> The **LTL** model checking problem is ***PSPACE***-complete

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  *space requirements:*

  recursion depth $\widehat{=}$ height of computation tree

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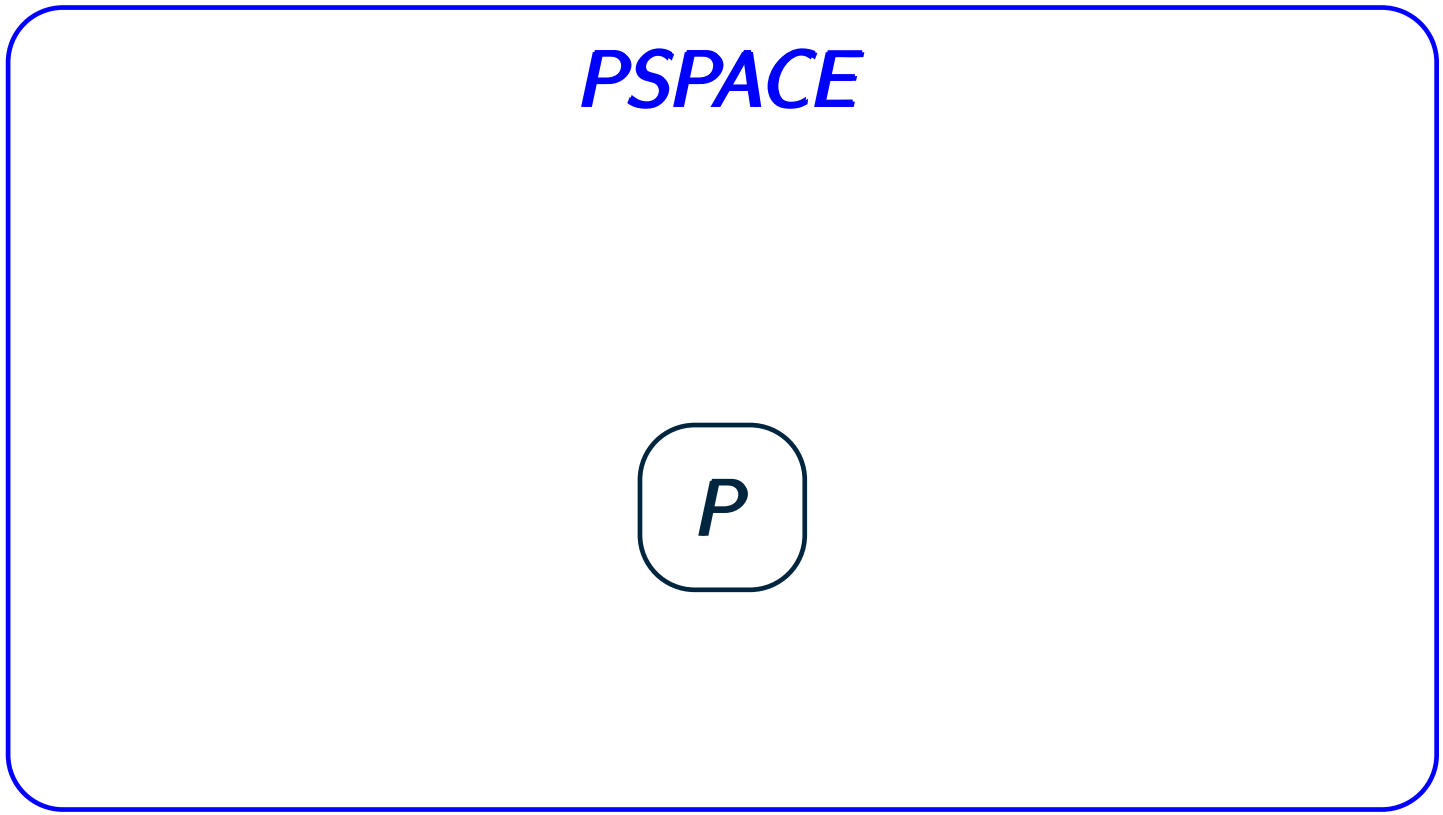
To prove $L \in$ ***PSPACE*** it suffices to provide a nondeterministic polynomially space-bounded algorithm for the complement $\overline{L}$ of $L$

# Complexity classes *P*, *NP*, *coNP*, *PSPACE*



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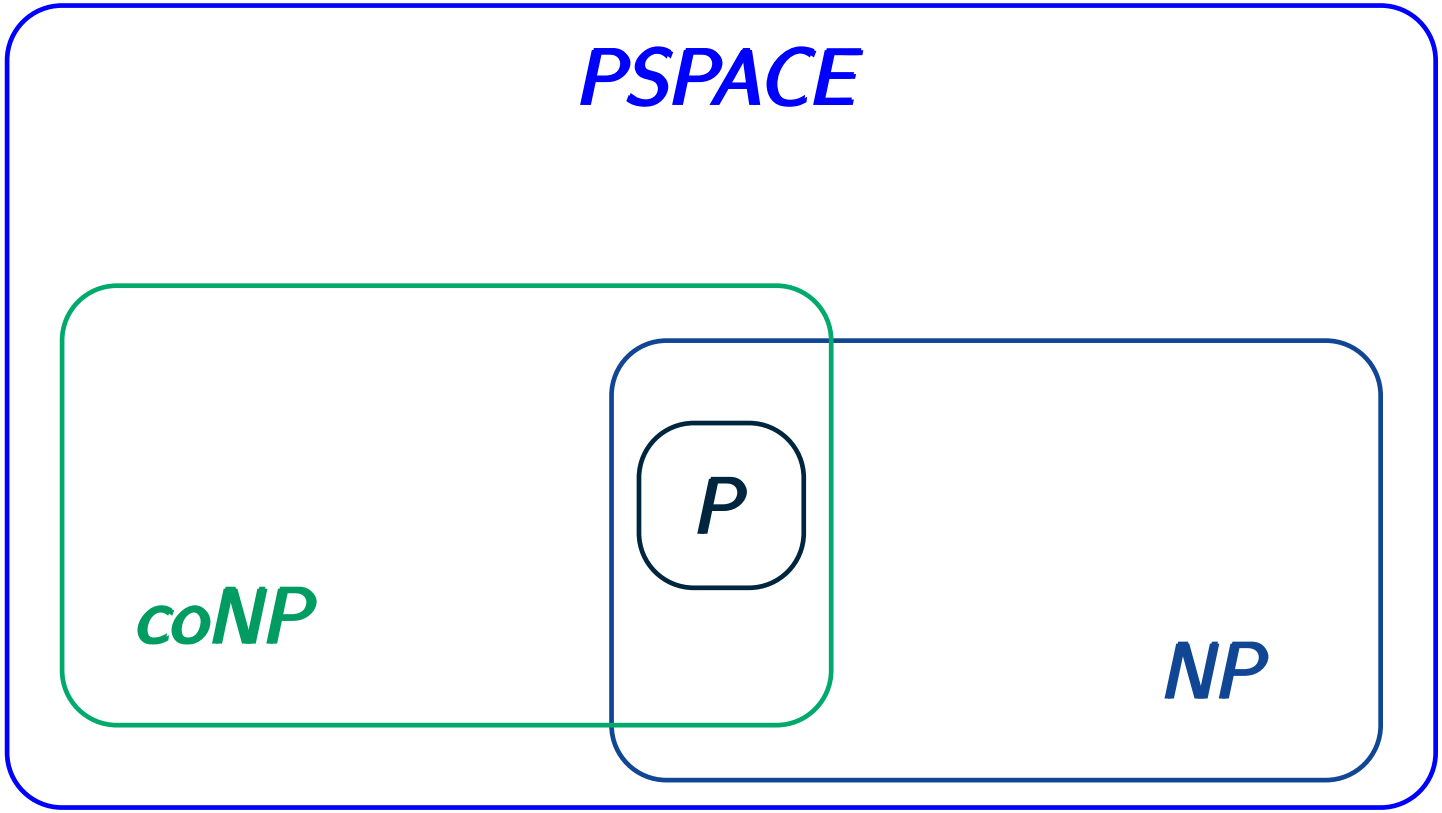


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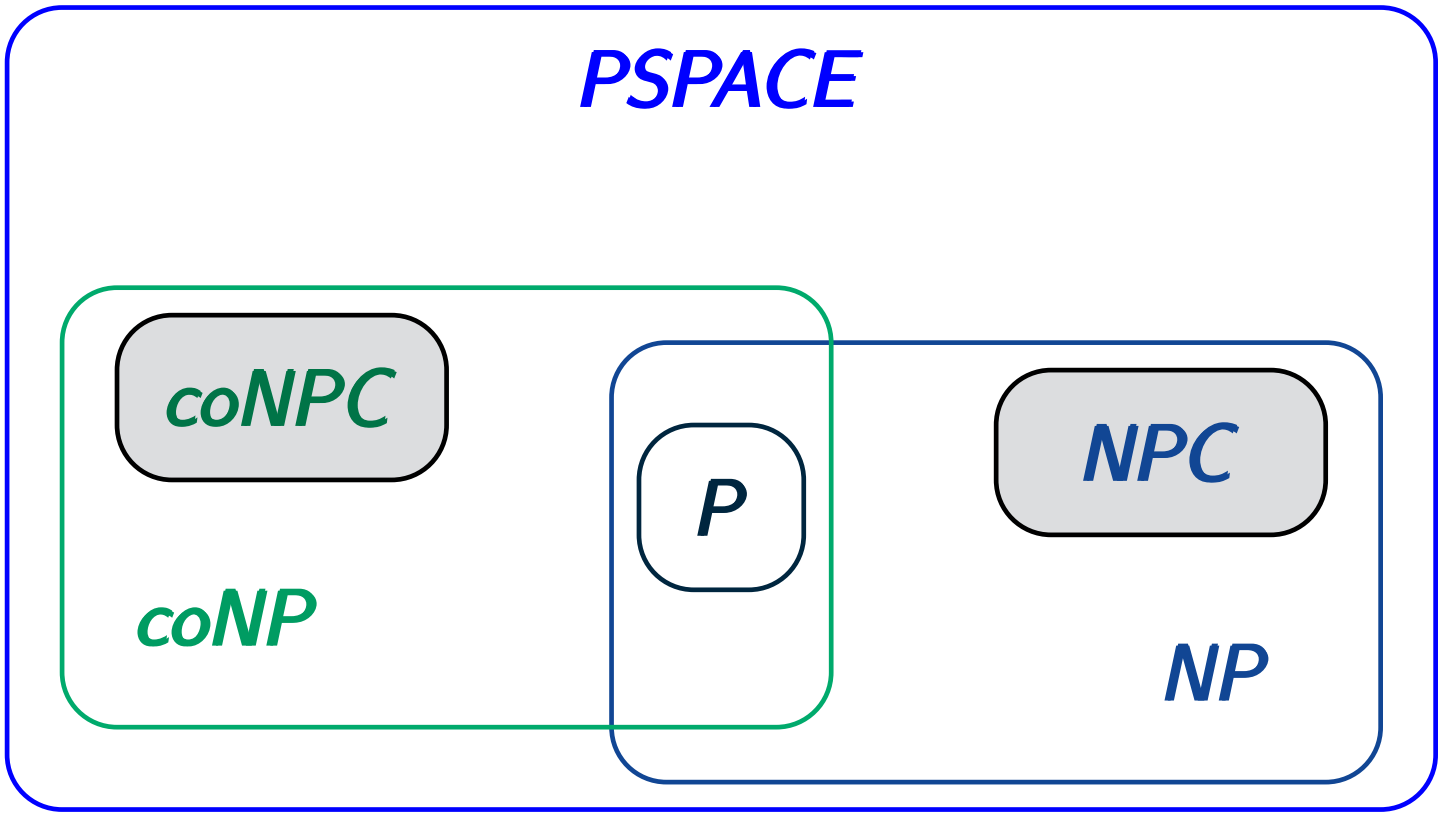


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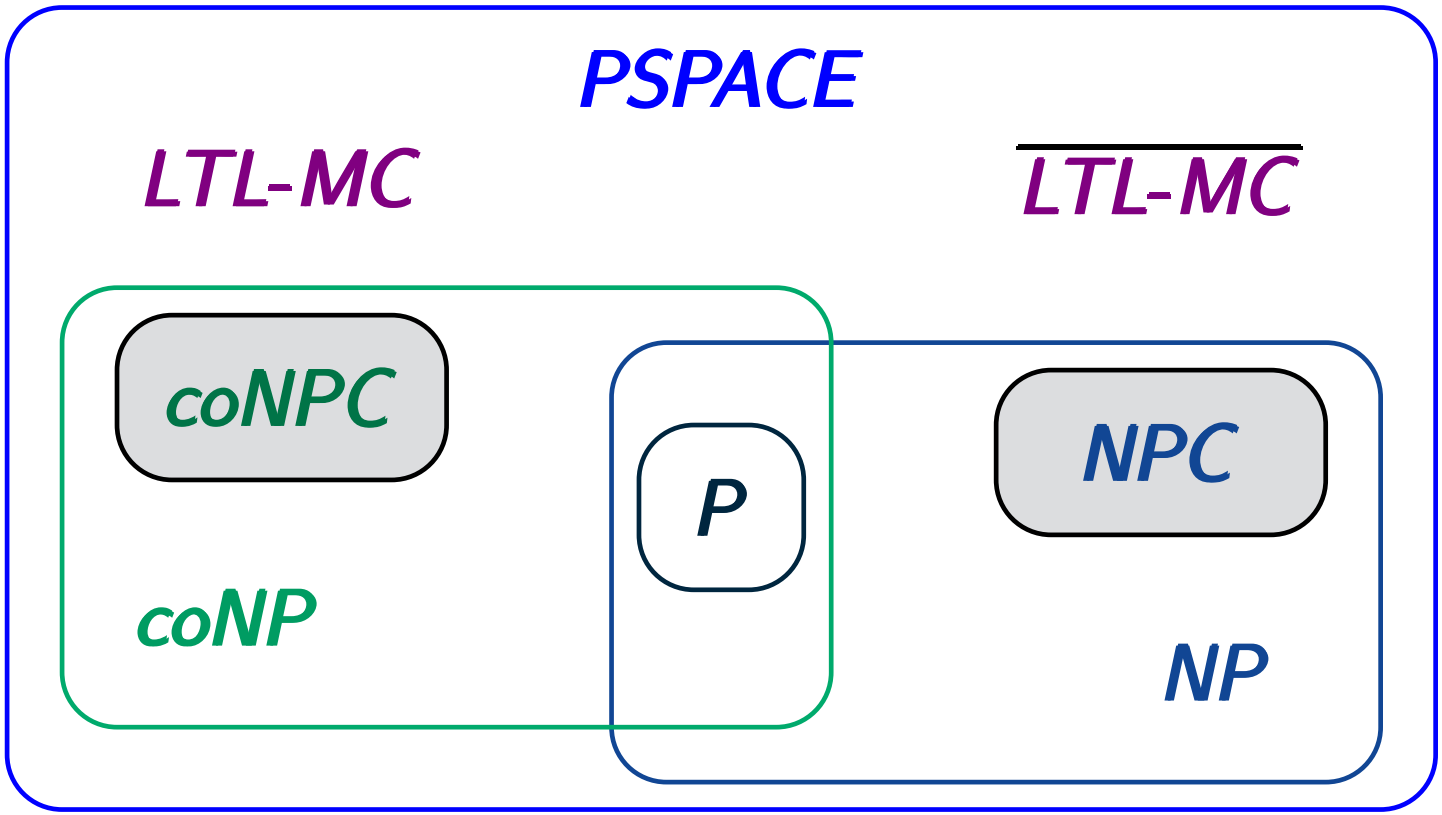


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