General Picture of LTL Model Checking with Büchi Automata

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Topics

- Automata-based LTL model checking. General picture.
- From LTL formulas to NBAs. Examples.
- NFA and NBA for safety properties.
- Examples of LTL model checking with NBA.
- LTL model checking complexity (without proof).

Material

Reading:

Chapter 5 of the book, Section 5.2.

More:

The slides in the following pages are taken from the material of the course "Introduction to Model Checking" held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

Overview

LTL model checking problem

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given: finite transition system $\mathcal{T}$ over *AP*
(without terminal states)
LTL-formula φ over *AP*

question: does $\mathcal{T} \models \varphi$ hold **?**

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$$\pi \not\models \varphi$$

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$$\pi \not\models \varphi, \text{ i.e., } \pi \models \neg\varphi$$

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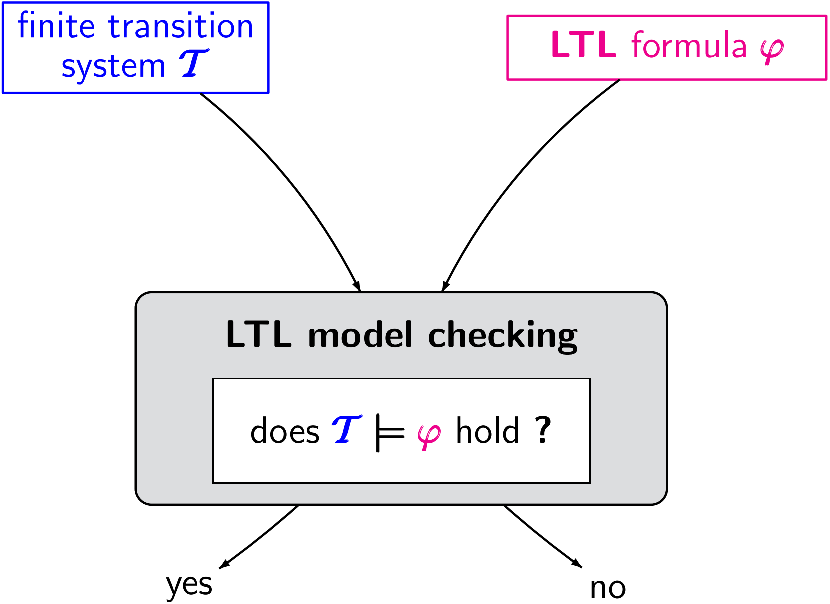
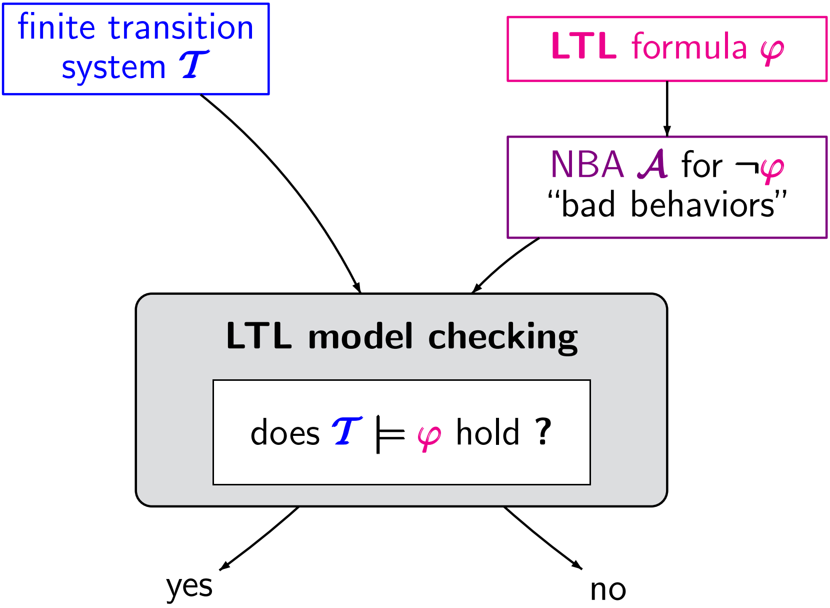
↑

construct the product-TS $\mathcal{T} \otimes \mathcal{A}$
search a path in the product that meets the acceptance condition of $\mathcal{A}$



finite transition system $\mathcal{T}$
LTL formula φ
LTL model checking
does $\mathcal{T} \models \varphi$ hold ?
yes
no

Automata-based LTL model checking



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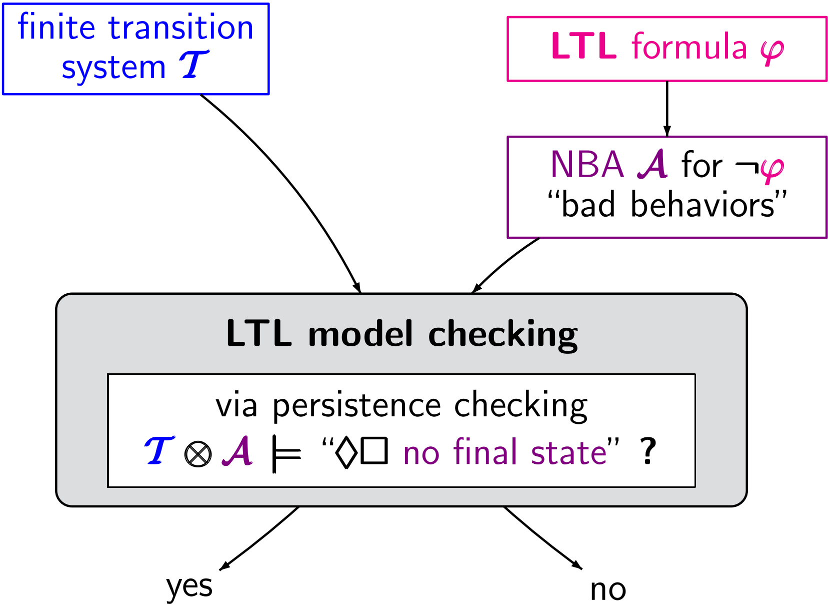

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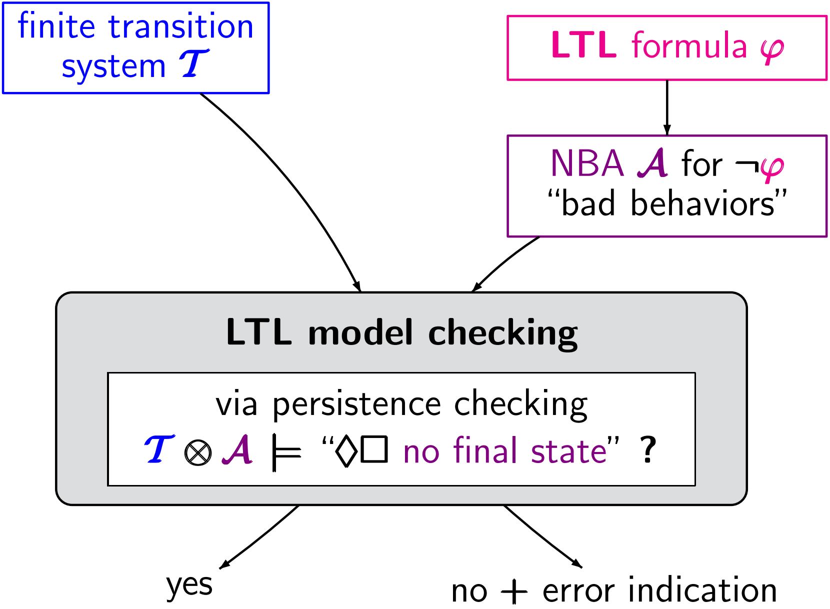

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safety property E	LTL-formula φ

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Safety vs LTL model checking

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$\mathcal{T} \models$ safety property E

iff $Traces_{fin}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}) = \varnothing$

where $\mathcal{A}$ is an NFA for the bad prefixes

$\mathcal{T} \models$ LTL-formula φ

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where $\mathcal{A}$ is an NBA for $\neg\varphi$

Safety vs LTL model checking

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iff $\mathcal{T} \otimes \mathcal{A} \models \Box \neg F$ $\longleftarrow$ invariant checking

---

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iff $\mathcal{T} \otimes \mathcal{A} \models \Diamond \Box \neg F$ $\longleftarrow$ persistence checking

Recall: nondeterministic Büchi automata

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

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run for a word $A_0 A_1 A_2 \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0 q_1 q_2 \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run π is accepting if $\overset{\infty}{\exists} i \in \mathbb{N}.\ q_i \in F$

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accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

$\mathcal{L}_\omega(\mathcal{A}) \overset{\text{def}}{=}$ set of infinite words over Σ that have an accepting run in $\mathcal{A}$

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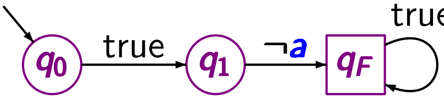
proof: ... later ...

NBA for LTL formulas



$\mathcal{L}_\omega(\mathcal{A}) = \textbf{?}$

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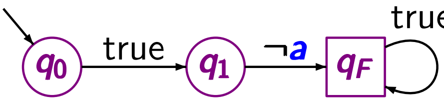

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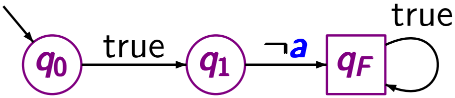

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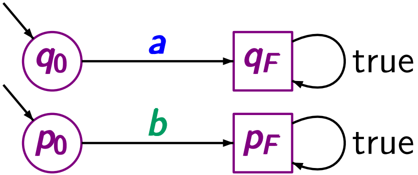

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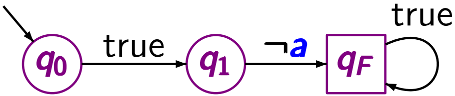

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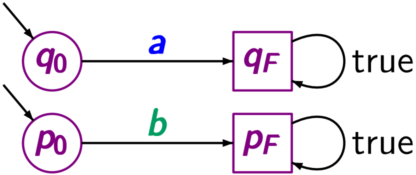

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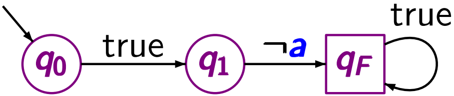

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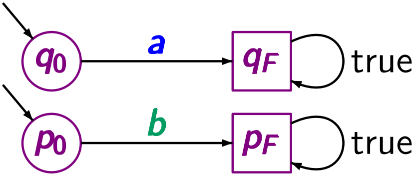

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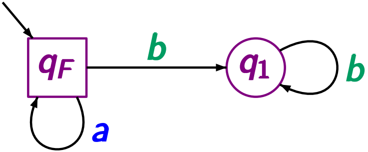

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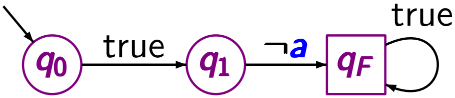

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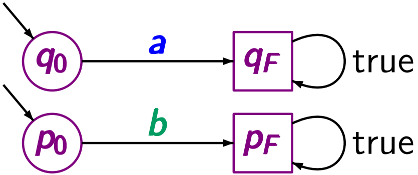

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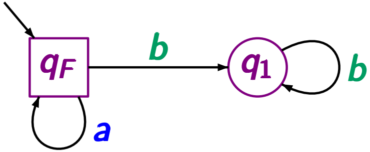

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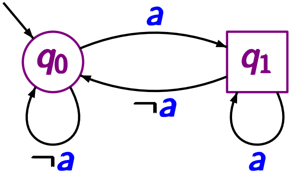
NBA for LTL formulas



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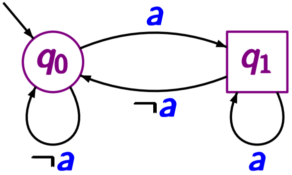

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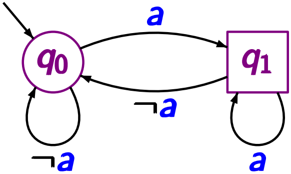

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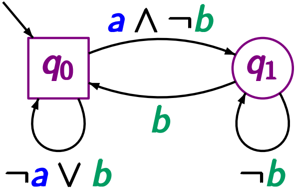

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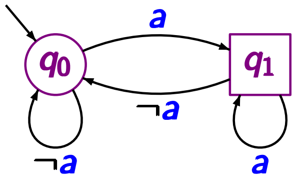

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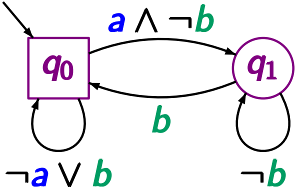

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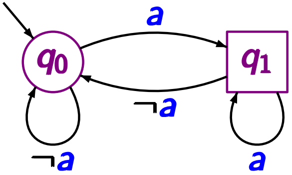
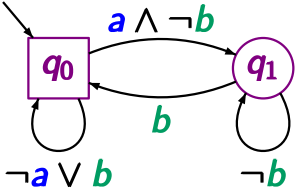
e.g., $\varnothing\,\varnothing\,\varnothing\,\varnothing \ldots = \varnothing^\omega$ and $(\{a\}\,\{b\})^\omega$ are accepted by $\mathcal{A}$



$$\mathcal{L}_\omega(\mathcal{A}) = \mathit{Words}(\Box\Diamond a)$$



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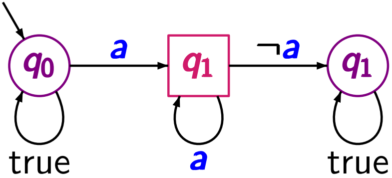
NBA for LTL formula



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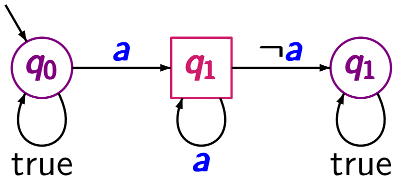

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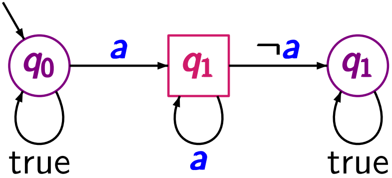

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possible runs for $\{a\}^\omega$

$q_0\ q_0\ q_0\ q_0\ q_0\ q_0\ \ldots$	not accepting
$q_0\ q_1\ q_1\ q_1\ q_1\ q_1\ \ldots$	accepting
$q_0\ q_0\ q_1\ q_1\ q_1\ q_1\ \ldots$	accepting
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$\vdots$

NFA and NBA for safety properties

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Let $\mathcal{A}$ be an **NFA** for the language of all bad prefixes for a safety property E.

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$$\mathcal{L}_\omega(\mathcal{A}) = \overline{E} = \left(2^{AP}\right)^\omega \setminus E$$

NFA and NBA for safety properties

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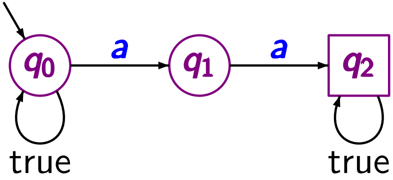
$$\mathcal{L}_\omega(\mathcal{A}) = \overline{E} = \left(2^{AP}\right)^\omega \setminus E$$

Example: $E \mathrel{\widehat{=}}$ "never a twice in a row"



NFA and NBA for safety properties

Let $\mathcal{A}$ be an **NFA** for the language of all bad prefixes for a safety property E. Then:

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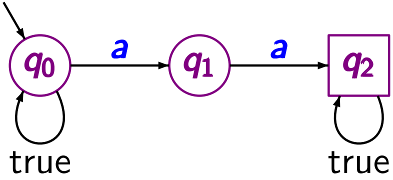

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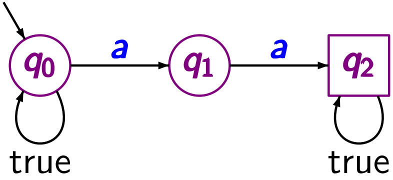

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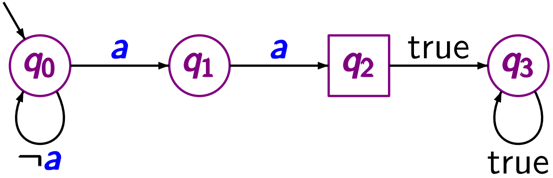

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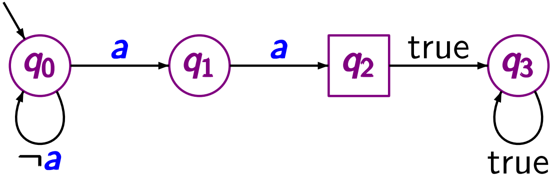

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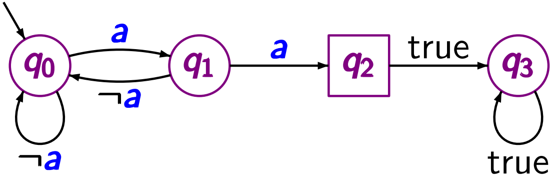

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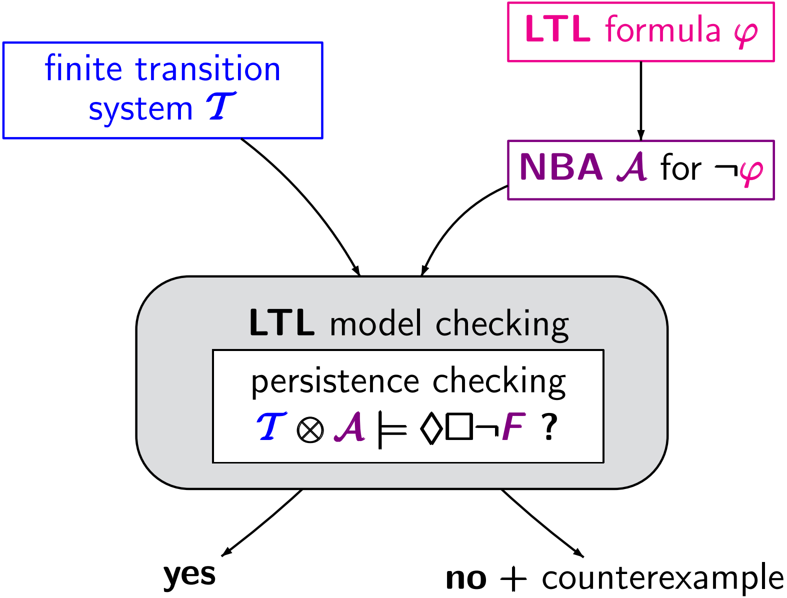
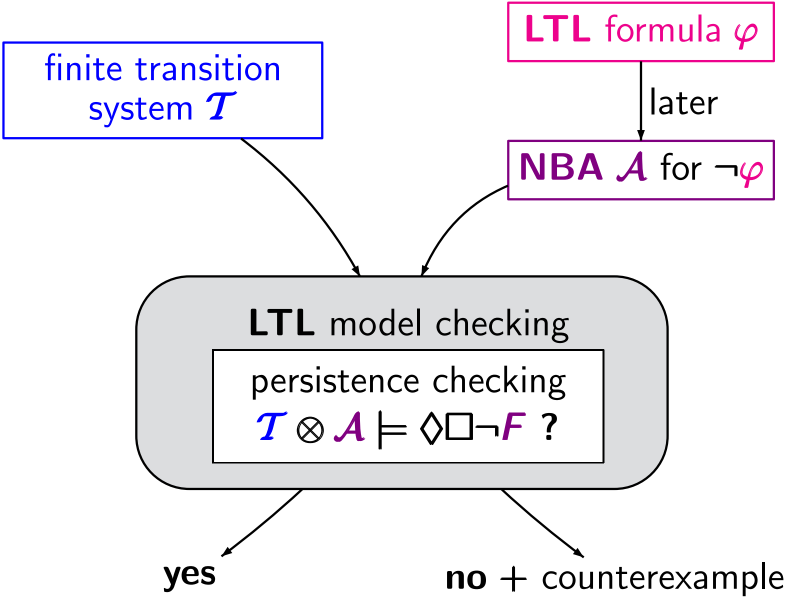
LTL model checking




finite transition system $\mathcal{T}$
LTL formula φ
later
NBA $\mathcal{A}$ for $\neg\varphi$
LTL model checking
persistence checking
$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$?
yes
no + counterexample

Recall: product transition system

$\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ TS without terminal states

$\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, F)$ NBA or NFA

non-blocking, $Q_0 \cap F = \varnothing$

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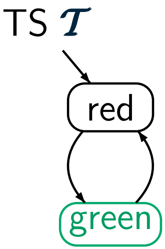
transition relation:

$$\frac{s \xrightarrow{\alpha} s' \ \wedge \ q' \in \delta(q, L(s'))}{\langle s, q\rangle \xrightarrow{\alpha}{}' \langle s', q'\rangle}$$

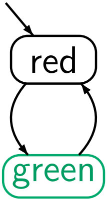
Example: LTL model checking



LTL formula $\varphi = \Box\Diamond \text{green}$

Example: LTL model checking

TS $\mathcal{T}$



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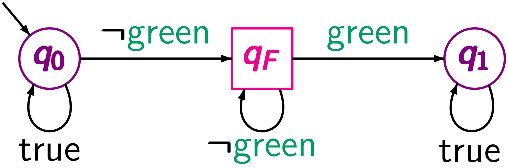
$\neg\varphi \equiv \Diamond\Box\neg\text{green}$



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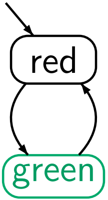

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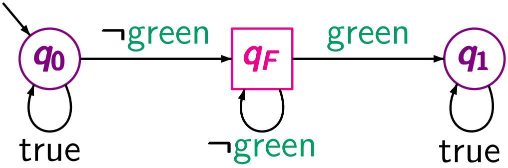
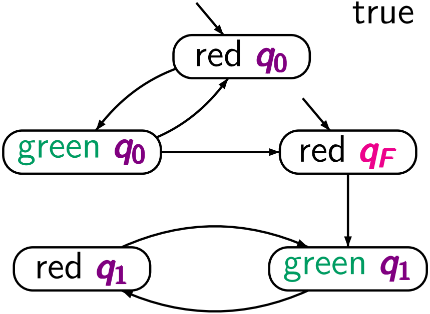
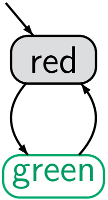




reachable fragment of the product TS $\mathcal{T} \otimes \mathcal{A}$

Example: LTL model checking

TS $\mathcal{T}$



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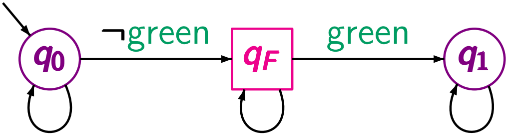
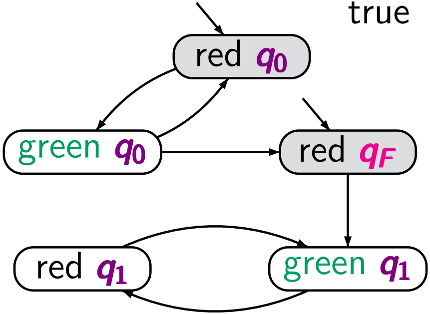


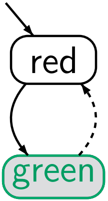


initial states:

$\langle red, q\rangle$ where

$$q \in \delta(q_0, L(red))$$
$$= \delta(q_0, \varnothing)$$
$$= \{q_0, q_F\}$$

TS $\mathcal{T}$



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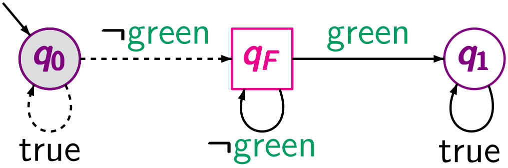
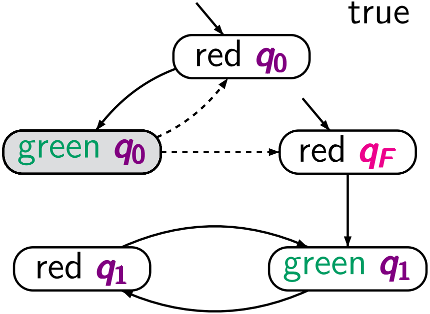




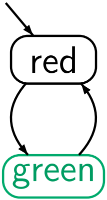
transition

$\langle \text{green}, q_0 \rangle \rightarrow \langle \text{red}, q \rangle$

$$q \in \delta(q_0, L(\text{red}))$$
$$= \delta(q_0, \varnothing)$$
$$= \{q_0, q_F\}$$

Example: LTL model checking

TS $\mathcal{T}$



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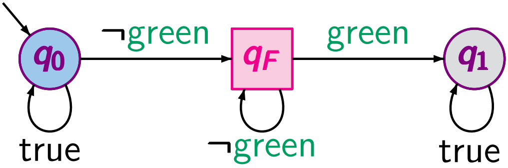
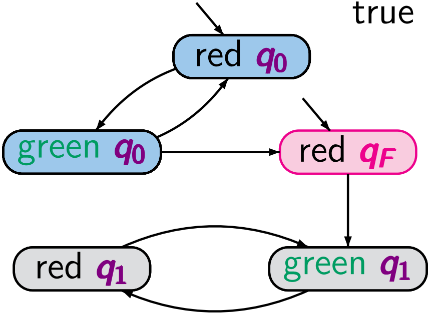




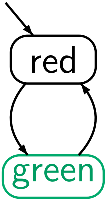
atomic propositions

$AP' = \{q_0, q_F, q_1\}$

obvious labeling function

Example: LTL model checking

TS $\mathcal{T}$



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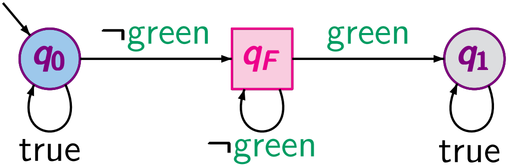

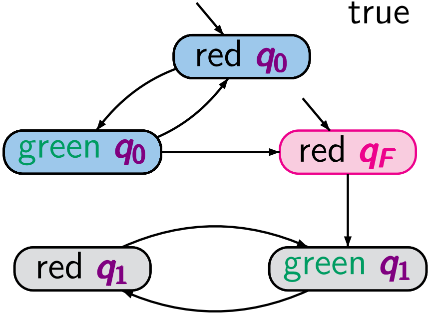

$\mathcal{T} \otimes \mathcal{A} \models \Diamond\Box\neg F$

Example: LTL model checking

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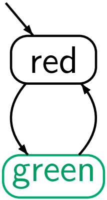

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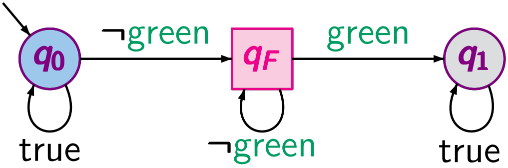

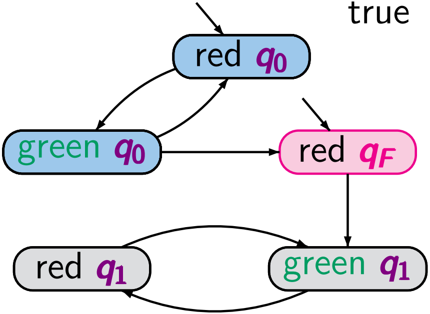

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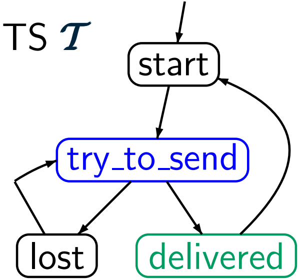
Example: LTL model checking



LTL formula $\varphi = \Box(\mathit{try} \rightarrow \Diamond \mathit{del})$

"each (repeatedly) sent message will eventually be delivered"

Example: LTL model checking

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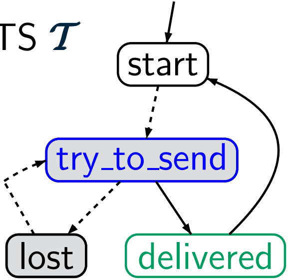

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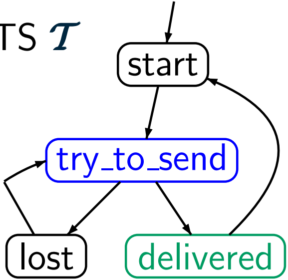
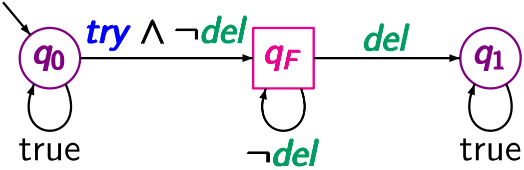


NBA $\mathcal{A}$ for $\neg\varphi \equiv \Diamond(try \wedge \Box\neg del)$



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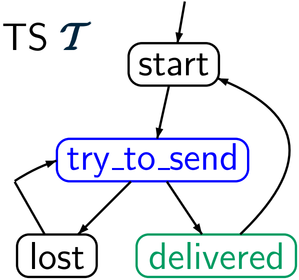

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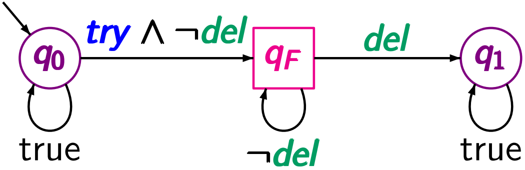
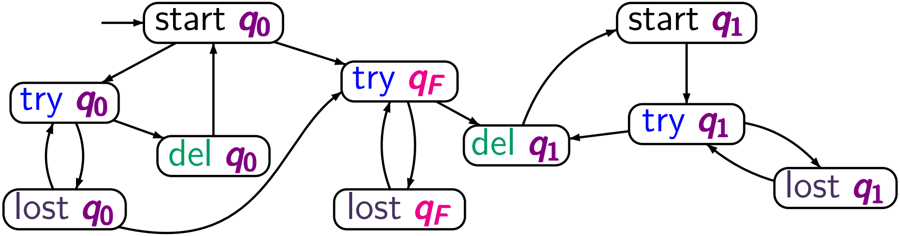
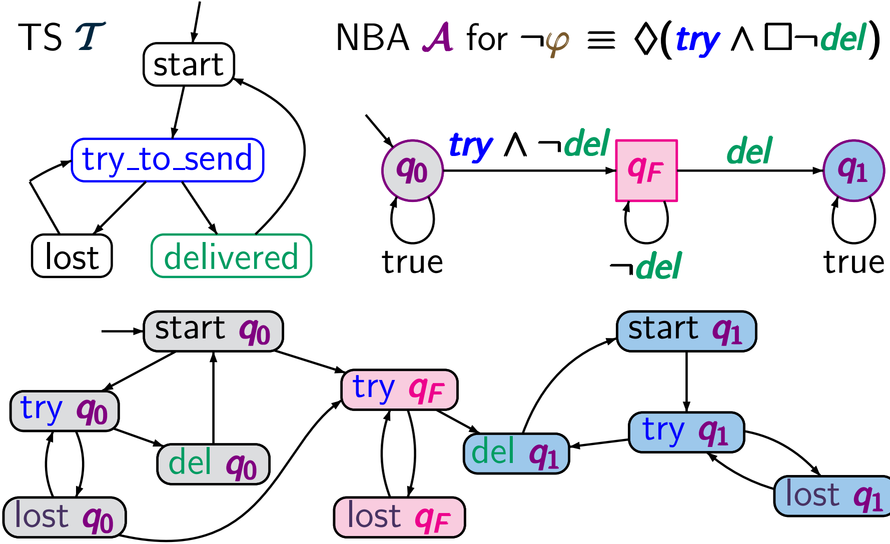




reachable fragment of the product-TS

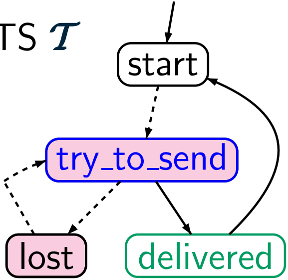
Example: LTL model checking



set of atomic propositions $AP' = \{q_0, q_1, q_F\}$

Example: LTL model checking

TS $\mathcal{T}$



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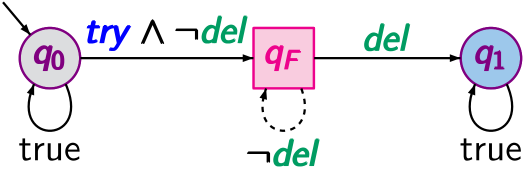
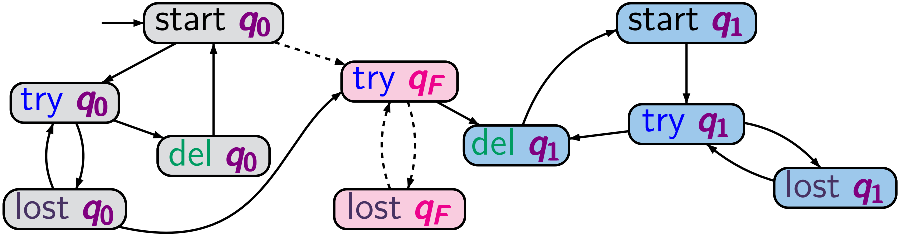
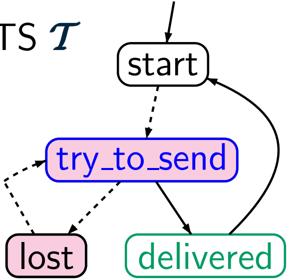




$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$

Example: LTL model checking

TS $\mathcal{T}$



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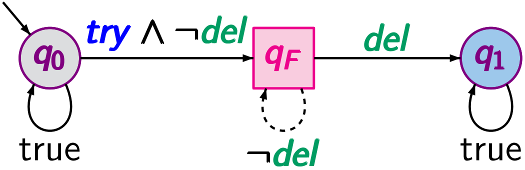

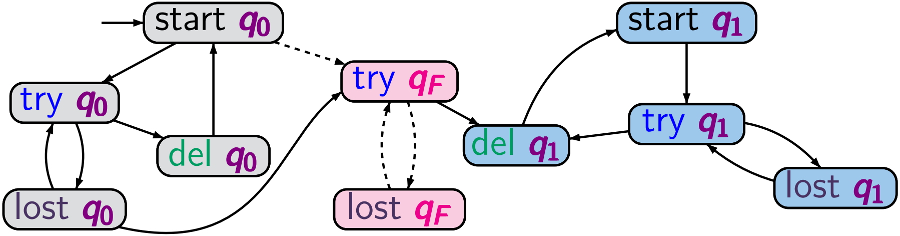

$\mathcal{T} \otimes \mathcal{A} \not\models \Diamond\Box\neg F$ hence: $\mathcal{T} \not\models \varphi$

Complexity of LTL model checking

main steps of automata-based LTL model checking:

construction of an NBA $\mathcal{A}$ for $\neg\varphi$

persistence checking in the product $\mathcal{T} \otimes \mathcal{A}$

Complexity of LTL model checking

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construction of an NBA $\mathcal{A}$ for $\neg\varphi$	$\longleftarrow \mathcal{O}(\exp(\lvert\varphi\rvert))$
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complexity: $\mathcal{O}(\mathit{size}(\mathcal{T}) \cdot \exp(\lvert\varphi\rvert))$

Complexity of LTL model checking

main steps of automata-based LTL model checking:

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complexity: $\mathcal{O}(size(\mathcal{T}) \cdot \exp(\lvert\varphi\rvert))$

The **LTL** model checking problem is **PSPACE**-complete

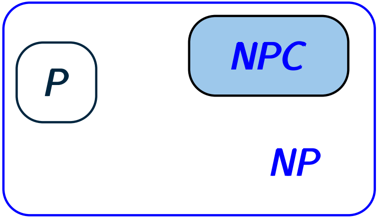
Recall: complexity classes

Complexity classes P, NP

P = class of decision problem solvable in deterministic polynomial time

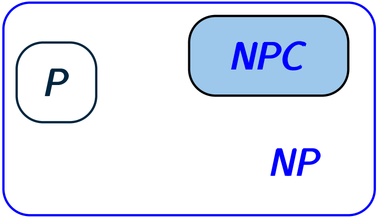
NP = class of decision problem solvable in nondeterministic polynomial time

Complexity classes *P*, *NP*



NPC = class of ***NP***-complete problems

Complexity classes *P*, *NP*

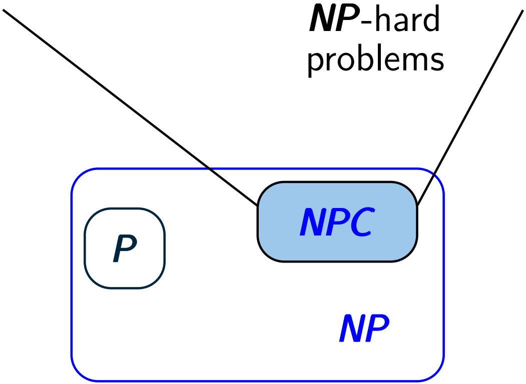


NPC = class of ***NP***-complete problems

(1) $L \in NP$

(2) L is ***NP***-hard, i.e., $K \leq_{poly} L$ for all $K \in NP$

Complexity classes P, NP



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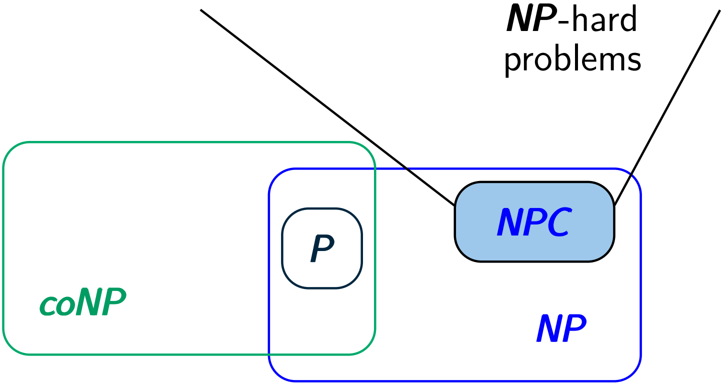
(2) L is NP-hard, i.e., $K \leq_{poly} L$ for all $K \in NP$

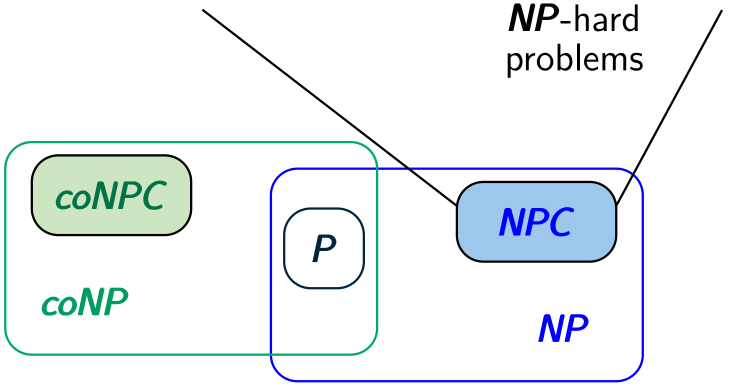
Complexity classes *P*, *NP*, *coNP*



$$coNP = \{ \overline{L} : L \in NP \}$$

↑ complement of L

Complexity classes *P*, *NP*, *coNP*



coNPC = class of ***coNP***-complete problems

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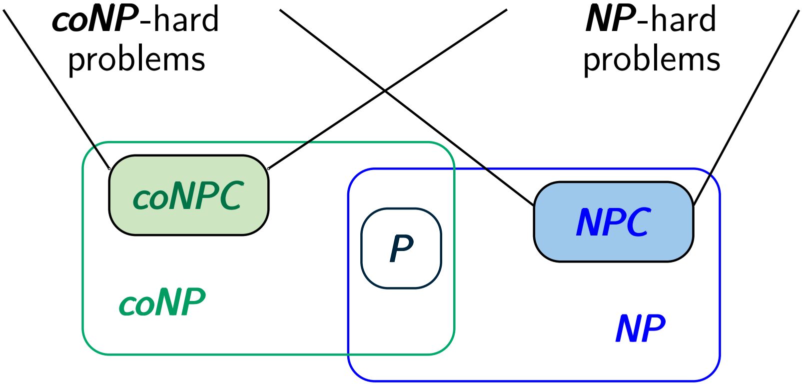
Complexity classes *P*, *NP*, *coNP*



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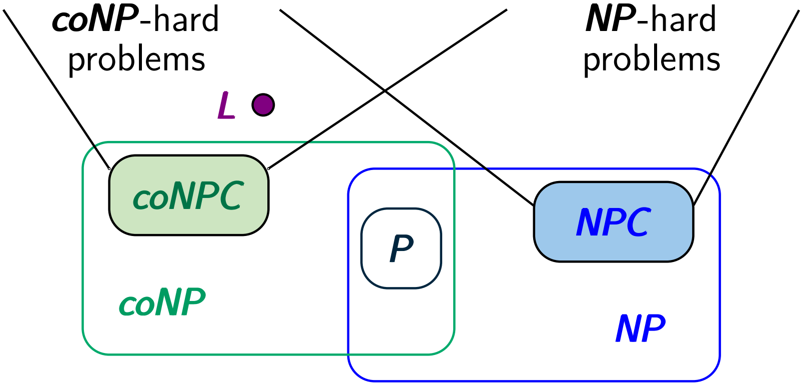
Complexity classes *P*, *NP*, *coNP*



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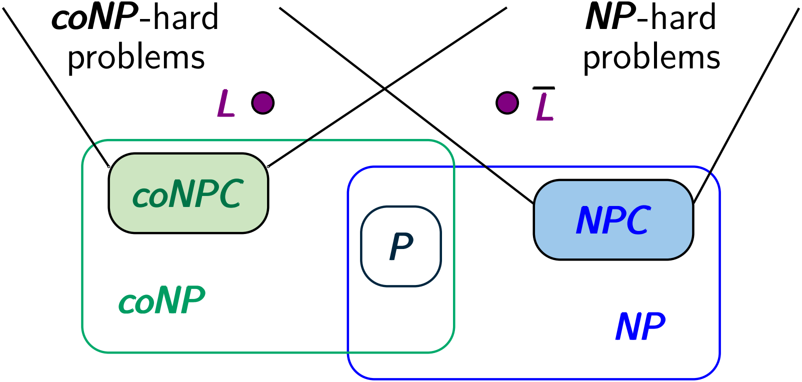

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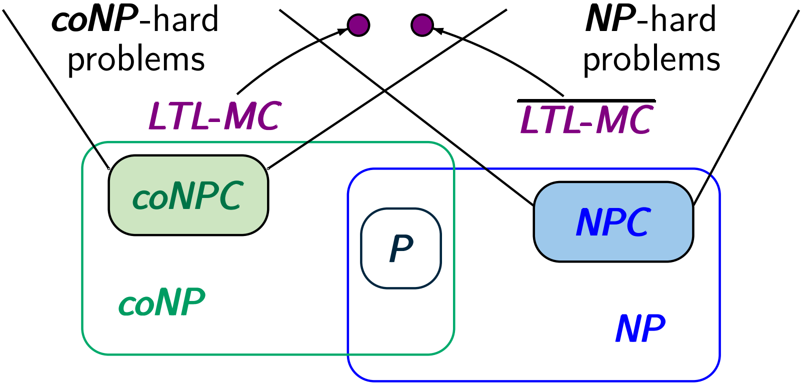

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***coNP*-hardness**

The **LTL** model checking problem is ***coNP***-hard

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proof by a polynomial reduction

$$HP \leq_{poly} \overline{LTL\text{-}MC}$$

HP: Hamilton path problem

$\overline{LTL\text{-}MC}$: complement of the **LTL** model checking problem

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complement of the **LTL** model checking problem:

given: finite transition system $\mathcal{T}$, LTL-formula φ

question: does $\mathcal{T} \not\models \varphi$ hold ?

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Hamilton path problem ↗ *HP*

NP-complete ↑ Hamilton path problem

complement of the **LTL** model checking problem ↖ $\overline{LTL\text{-}MC}$

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coNP-hardness

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HP: Hamilton path problem — **NP**-complete

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complement of the **LTL** model checking problem:

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Complexity of LTL model checking

We just saw:

The **LTL** model checking problem is ***coNP***-hard

We now prove:

The **LTL** model checking problem is ***PSPACE***-complete

The complexity class *PSPACE*

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PSPACE = class of decision problems solvable by a deterministic polynomially space-bounded algorithm

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  ↑

  DFS-based analysis of the computation tree of an ***NP***-algorithm

PSPACE $=$ class of decision problems solvable by a deterministic polynomially space-bounded algorithm

- ***NP*** $\subseteq$ ***PSPACE***

  $\uparrow$

DFS-based analysis of the computation tree of an ***NP***-algorithm

space requirements:

recursion depth $\widehat{=}$ height of computation tree

The complexity class *PSPACE*

PSPACE = class of decision problems solvable by a deterministic polynomially space-bounded algorithm

- ***NP*** $\subseteq$ ***PSPACE***
- ***PSPACE*** = ***coPSPACE***

  (holds for any deterministic complexity class)

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  (holds for any deterministic complexity class)
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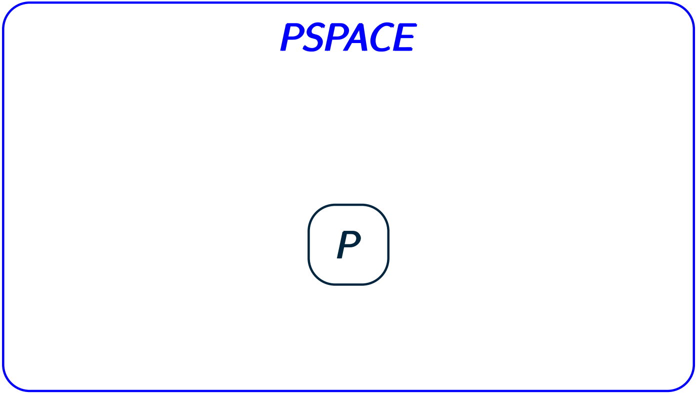
↑

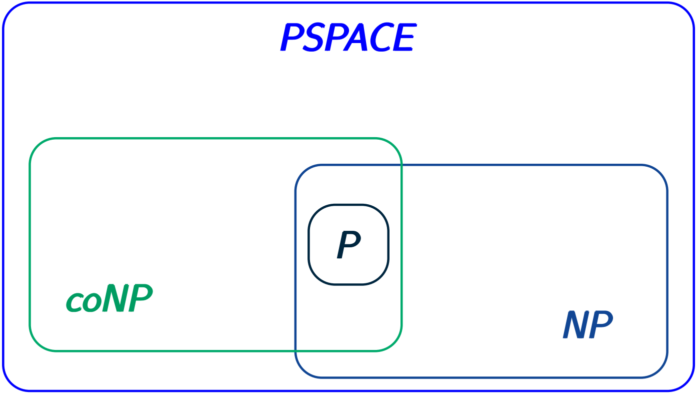
To prove $L \in$ *PSPACE* it suffices to provide a nondeterministic polynomially space-bounded algorithm for the complement $\overline{L}$ of L

Complexity classes P, NP, $coNP$, $PSPACE$



$PSPACE$ = class of decision problems that are decidable in deterministic polynomial space

Complexity classes *P*, *NP*, *coNP*, *PSPACE*



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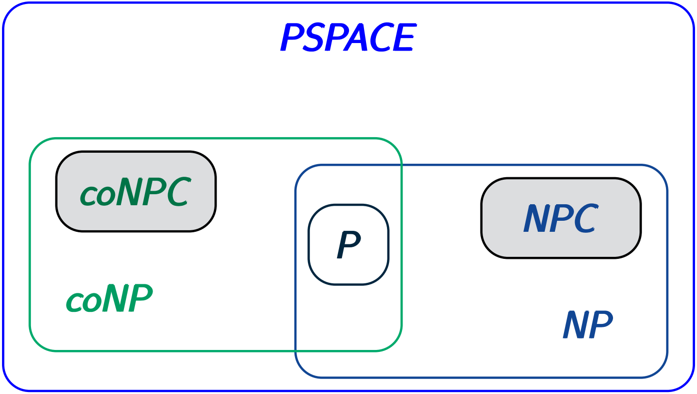

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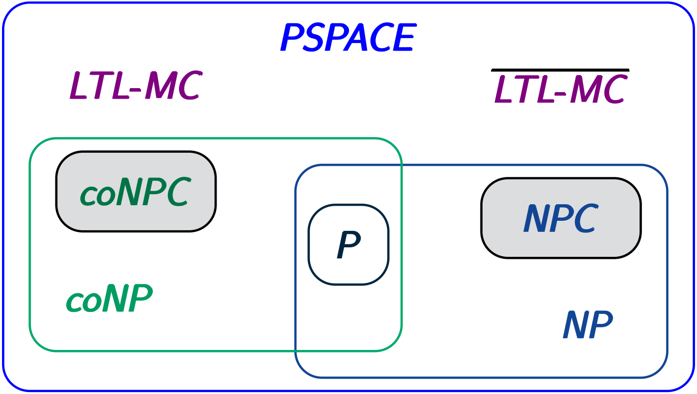

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