Overview

- Introduction
- Modelling parallel systems
- Linear Time Properties
- Regular Properties
- Linear Temporal Logic (LTL)
- Computation-Tree Logic
- **Equivalences and Abstraction**
  - bisimulation
  - CTL, CTL*-equivalence
  - computing the bisimulation quotient ⟵
  - abstraction stutter steps
  - simulation relations

Bisimulation quotient $\mathcal{T}/\sim$

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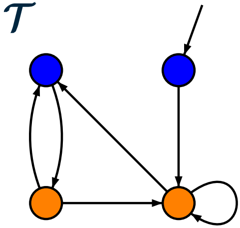

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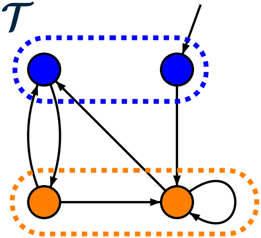

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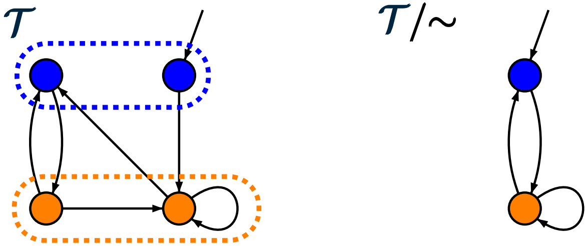

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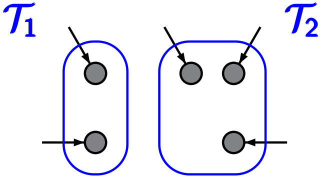

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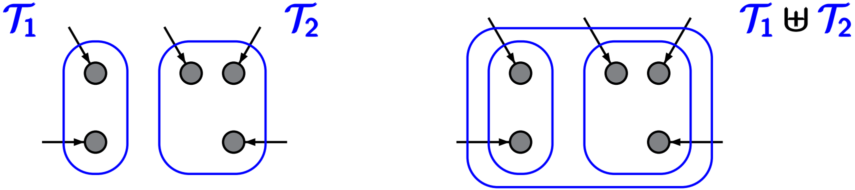

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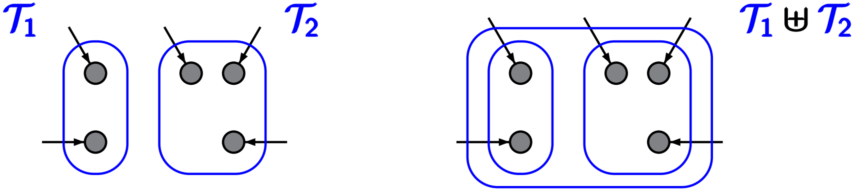

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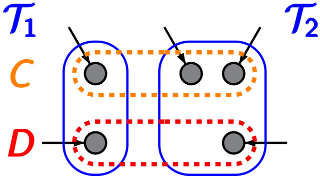

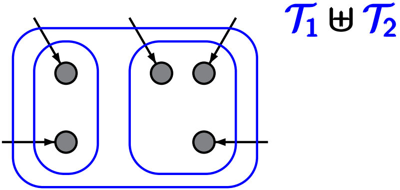

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2. graph minimization:

   replace $\mathcal{T}$ with $\mathcal{T}/\!\sim$ and analyze $\mathcal{T}/\!\sim$

Computing the bisimulation quotient

.... relies on a partitioning refinement algorithm ...

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here: only explanations for finite transition systems, possibly with terminal states

Notations: partitions and co.

PARTSPLITALG5.3-4

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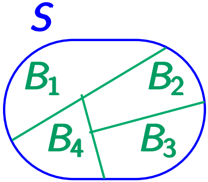

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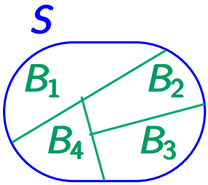

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- $S = B_1 \cup \ldots \cup B_k$

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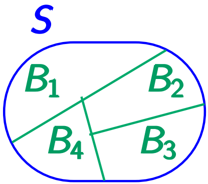

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A superblock denotes any union of blocks.

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- equivalence $\mathcal{R}$ on $S \rightsquigarrow$ partition $\mathcal{B} = S/\mathcal{R}$

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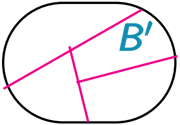

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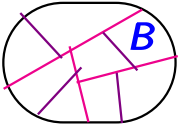

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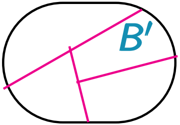

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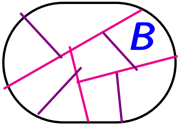

$\mathcal{B}_1$

Example: if $\mathcal{R}$ is a bisimulation for $\mathcal{T}$ and an equivalence then $S/\mathcal{R}$ is *finer* than $S/\sim$

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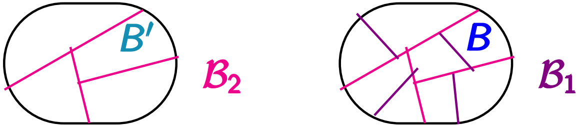

$\mathcal{B}_1$ is called *strictly finer* than $\mathcal{B}_2$ if

(1) $\mathcal{B}_1$ is finer than $\mathcal{B}_2$ and (2) $\mathcal{B}_1 \neq \mathcal{B}_2$

Computing the bisimulation quotient

by stepwise refinement of partitions the state set S

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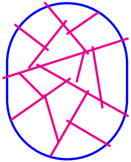

$S/\sim$

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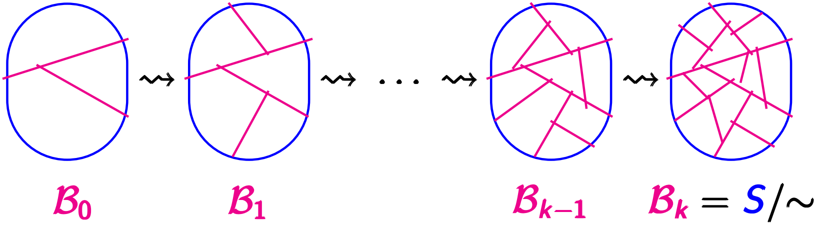

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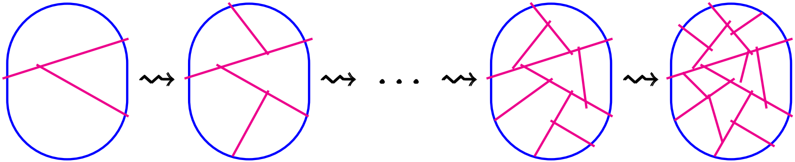

$\mathcal{B}_{AP} = \mathcal{B}_0 \quad \mathcal{B}_1 \quad \mathcal{B}_{k-1} \quad \mathcal{B}_k = S/\!\sim$

initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0$

identifies all states with the same labeling

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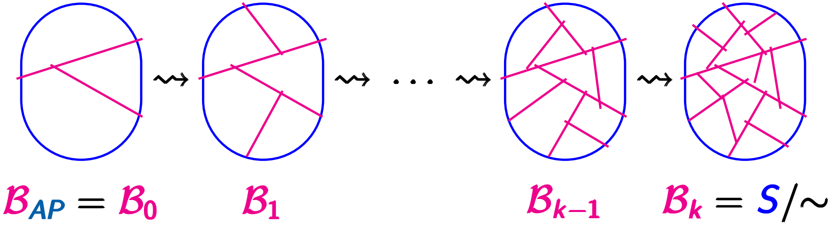

initial partition: $\mathcal{B}_{AP} = \mathcal{B}_0 = S/\mathcal{R}_{AP}$ where

$$\mathcal{R}_{AP} = \{ (s_1, s_2) : L(s_1) = L(s_2) \}$$

Characterization of $S/{\sim_{\mathcal{T}}}$

... as the coarsest partition of the
state space S such that ....

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where $Pre(C) = \{s \in S : \exists s' \in C \text{ s.t. } s \to s'\}$

Partitioning refinement algorithm

input: finite TS $\mathcal{T}$ with state space S over AP (possibly with terminal states)

output: bisimulation quotient $S/{\sim_{\mathcal{T}}}$

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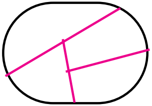
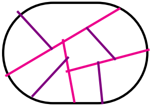
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

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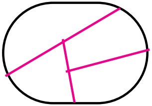

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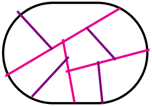

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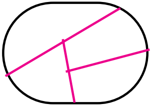
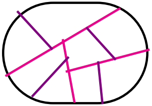
$\mathcal{B}_i$ 

$\mathcal{B}_{i+1}$ 

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UNTIL $\mathcal{B}_i = \mathcal{B}_{i-1}$ ← no more refinement possible, hence: $\mathcal{B}_i = S/\!\sim_{\mathcal{T}}$

return $\mathcal{B}_i$

$\mathcal{B}_i$  $\mathcal{B}_{i+1}$

loop invariant:

$\mathcal{B}_i$ is coarser than $S/\!\sim_{\mathcal{T}}$ and finer than $\mathcal{B}_{AP}$

Maximal number of iterations?

$\mathcal{B}_0 := \mathcal{B}_{AP}$; $i := 0$

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$\mathcal{B}_{i+1} := \mathit{Refine}(\mathcal{B}_i)$; $i := i+1$

UNTIL no further refinement is possible

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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps ?

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answer: $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

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Assuming that $\mathcal{B}_i$ is strictly coarser than $\mathcal{B}_{i+1}$ for all i, what is the maximal number of refinement steps ?

answer: $|S| - 1$

Note that $|\mathcal{B}_i| \geq i+1$.

Hence: if there are $k = |S|-1$ iterations then $\mathcal{B}_k$ consists of singletons

The initial partition

initial partition $\mathcal{B}_{AP}$:

identifies all states s, t s.t. $L(s) = L(t)$

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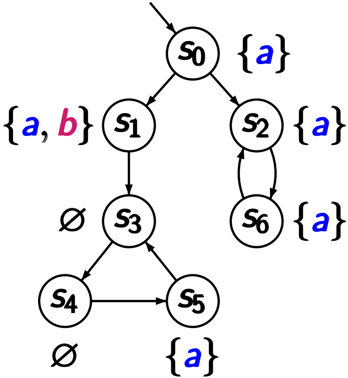

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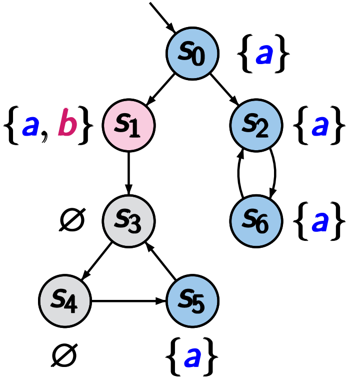

$$\mathcal{B}_{AP} = \left\{ \{s_0, s_2, s_6, s_5\}, \{s_1\}, \{s_3, s_4\} \right\}$$

Initial partition

initial partition $\mathcal{B}_{AP}$:

- identifies all states with the same labeling
- agrees with the quotient under the equivalence

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inner nodes at level i: decision "$a_i \in L(s)$?"

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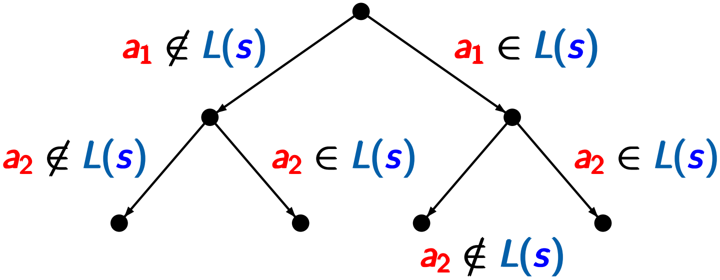

Computing the initial partition

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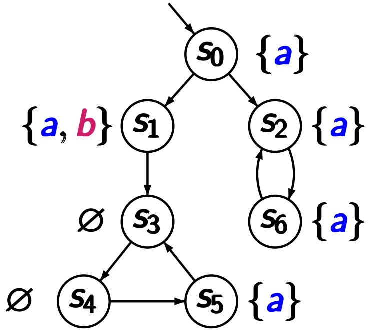
initally: each leaf represents the empty state-set

for each state s:

  traverse the decision tree from the root to a leaf v

  insert s in the set for v

Example: initial partition



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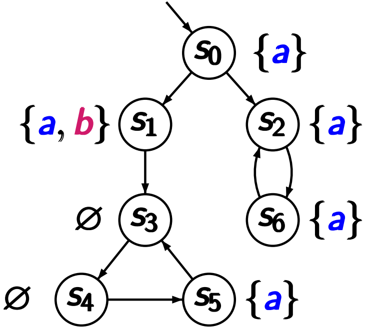

decision tree for
$AP = \{a, b\}$

1. level: $a \in L(s)$?
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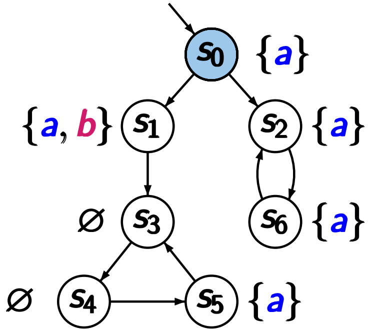
Example: initial partition



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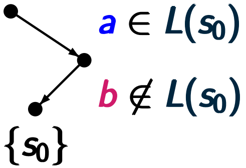

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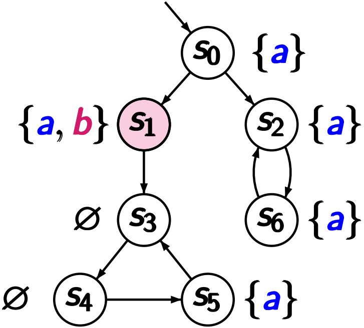

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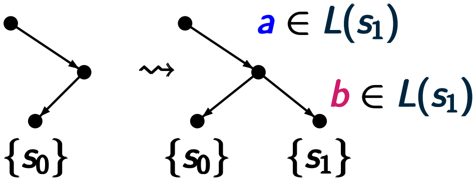

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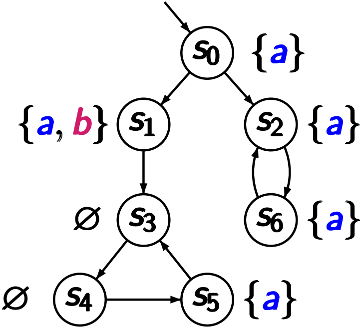

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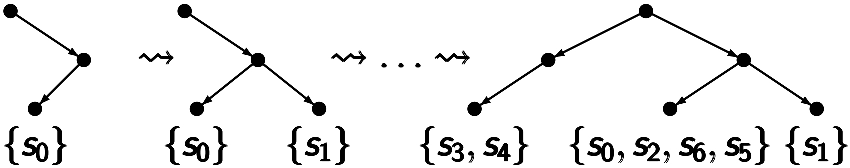

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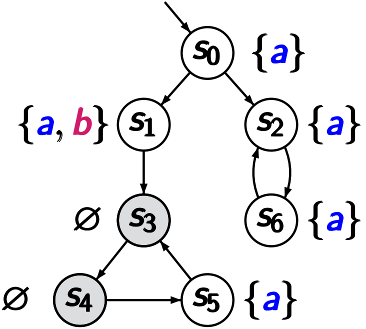

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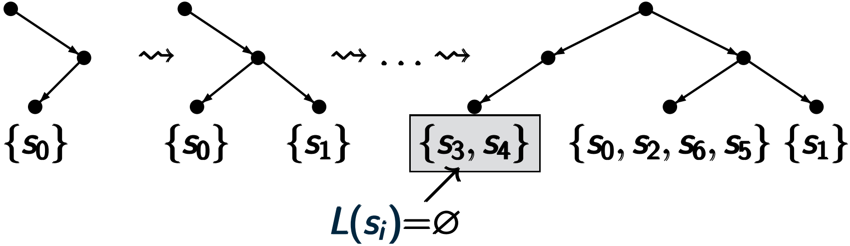

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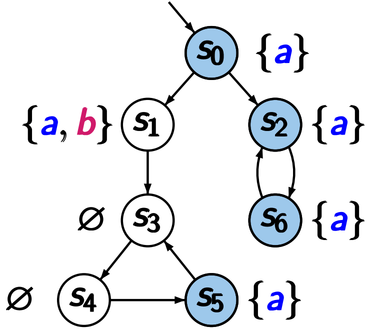

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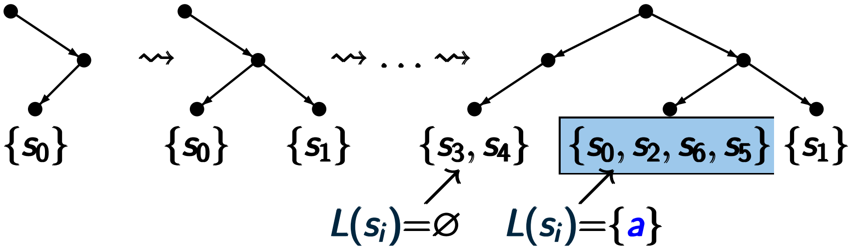

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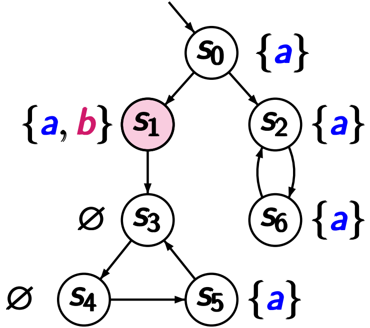

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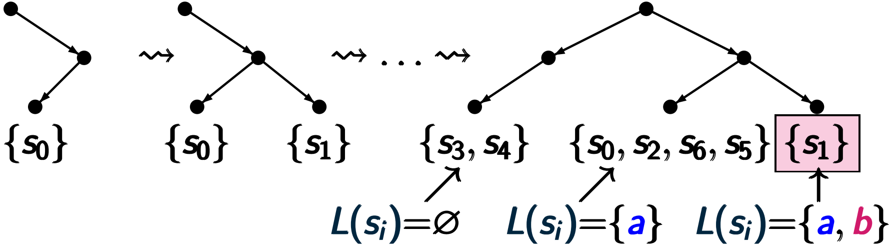

Computation of the initial partition

```
generate the root node v_0 of the decision tree
FOR ALL states s DO
   v := v_0
   FOR i = 1,...,k OD
      IF a_i ∈ L(s)
         THEN  v := find_or_add(right son of v)
         ELSE  v := find_or_add(left son of v)
      FI
   OD

OD
```

suppose
$AP = \{a_1, \ldots, a_k\}$

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   OD  ⟵ [v is a leaf of depth k]
   add s into the state-set of v
OD
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The state-sets of the leaves are the blocks in $\mathcal{B}_{AP}$.

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complexity:
$\mathcal{O}(|S| \cdot |AP|)$

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Partitioning refinement (schema)

$\mathcal{B} := \mathcal{B}_{AP}$

WHILE refinements are possible DO

$\qquad \mathcal{B} := \mathit{Refine}(\mathcal{B})$

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return $\mathcal{B}$

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split each block $B \in \mathcal{B}$ into two blocks:

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Partitioning refinement (schema)


```
𝓑 := 𝓑_AP

WHILE  refinements are possible DO

        𝓑 := Refine(𝓑, C) for some splitter C
OD

return 𝓑
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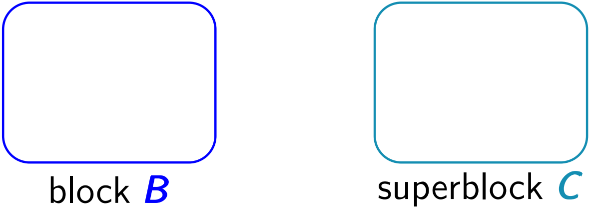
$\mathit{Refine}(B, C)$



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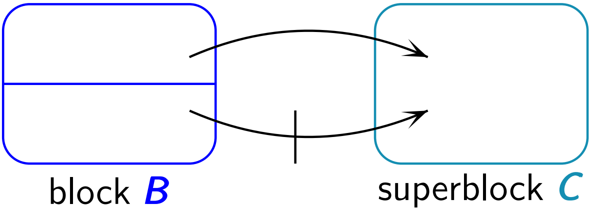

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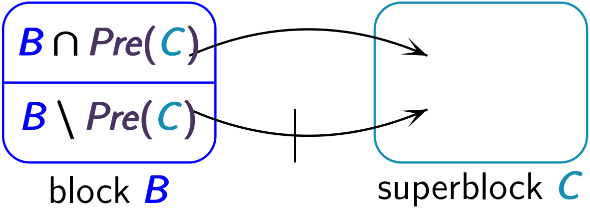
$$Refine(B, C) = \{B \cap Pre(C), B \setminus Pre(C)\}$$



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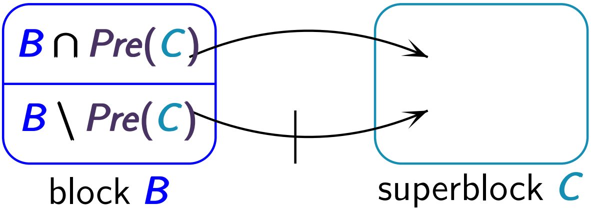

The refinement operator

Let $\mathcal{B}$ be a partition for S and C a superblock of $\mathcal{B}$.

$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

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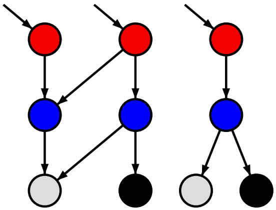
(c) $Refine(\mathcal{B}, C) = \mathcal{B}$ for all $C \in \mathcal{B}$ iff $\mathcal{B} = S/\sim_{\mathcal{T}}$

Example: partitioning splitter algorithm



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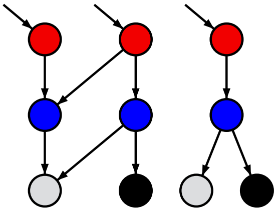

⟹

refinement w.r.t. ●

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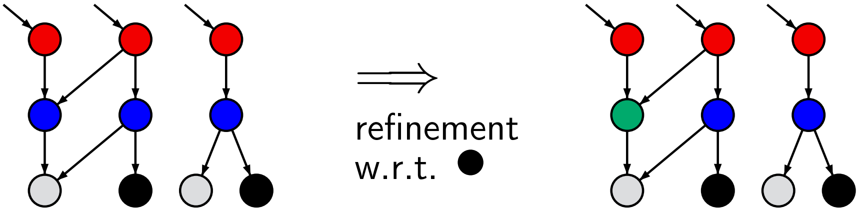

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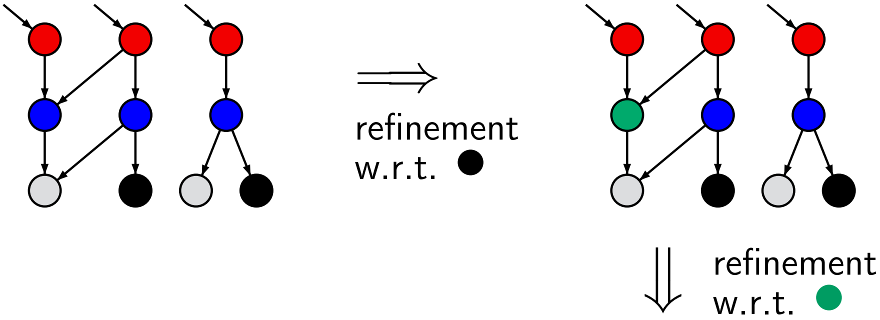

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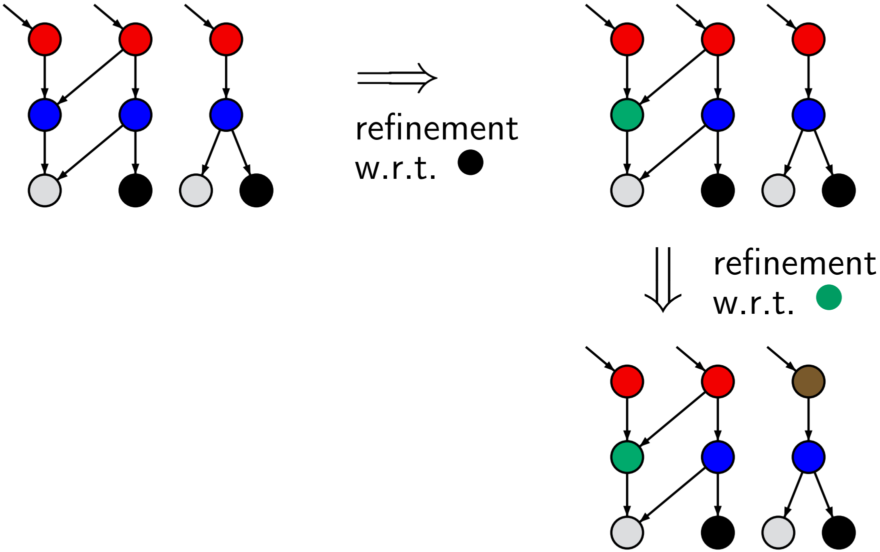

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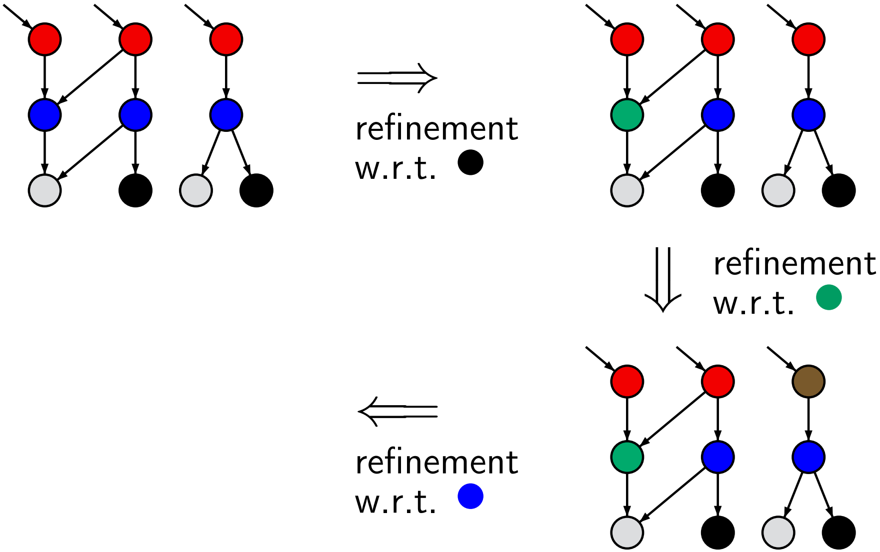

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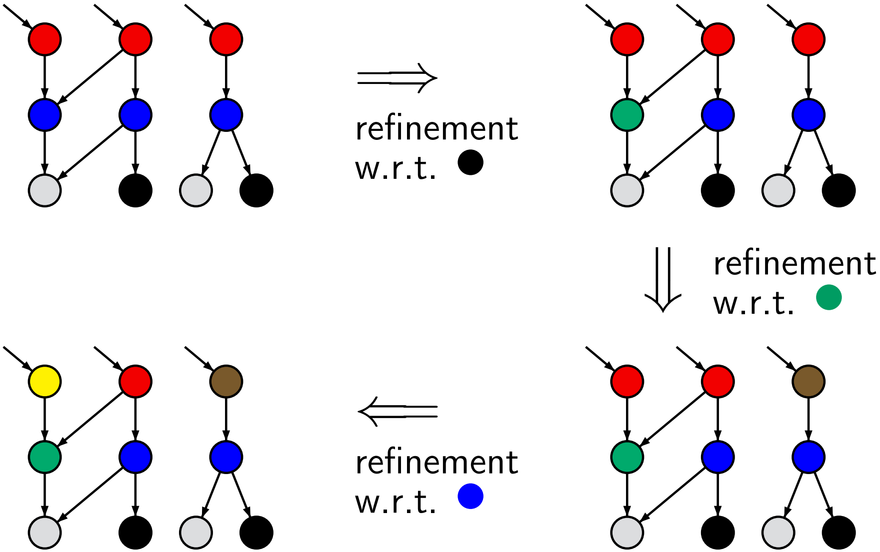

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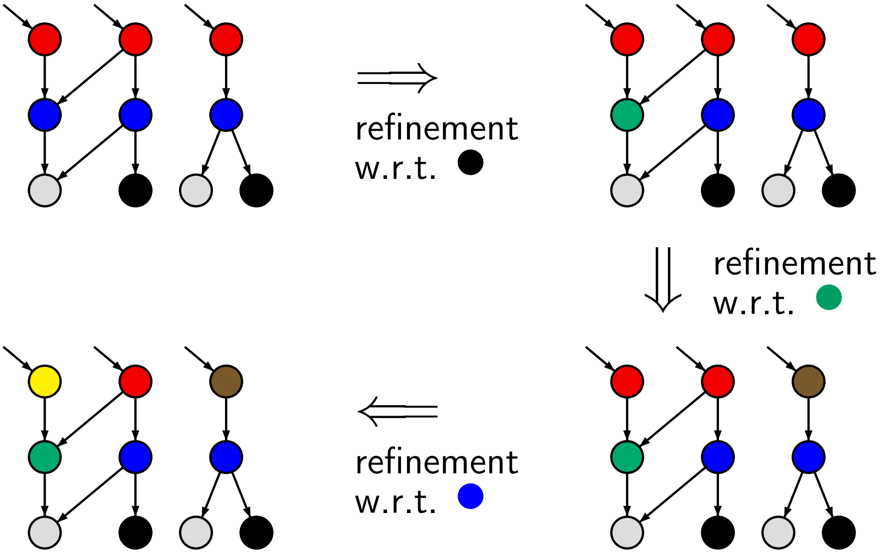

7 bisimulation equivalence classes

The refinement operator

given a partition $\mathcal{B}$ and a superblock C of $\mathcal{B}$,
how to compute

$$Refine(\mathcal{B}, C)$$

efficiently **?**

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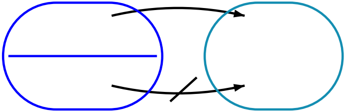
$$Refine(\mathcal{B}, C) = \bigcup_{B \in \mathcal{B}} Refine(B, C)$$

efficiently **?**

where for all blocks $B \in \mathcal{B}$:

$$Refine(B, C) = \{\, B \cap Pre(C),\ B \setminus Pre(C) \,\} \setminus \{\varnothing\}$$



block B superblock C

Refinement operator *Refine*($\mathcal{B}$, C)

Refinement operator $Refine(\mathcal{B}, C)$


```
FOR ALL s′ ∈ C DO



OD
```

Refinement operator $Refine(\mathcal{B}, C)$


```
FOR ALL s' ∈ C DO
   FOR ALL s ∈ Pre(s') DO



   OD
OD
```

Refinement operator $Refine(\mathcal{B}, C)$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   FOR ALL s ∈ Pre(s') DO
         "move" state s from block [s]_B = B
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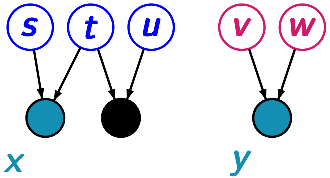
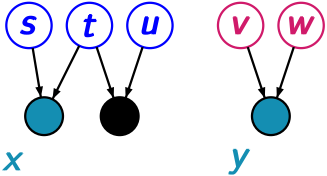
time complexity:

$$\mathcal{O}\Big(\sum_{s' \in C} |Pre(s')| + |C| \Big)$$

Example: refinement operator

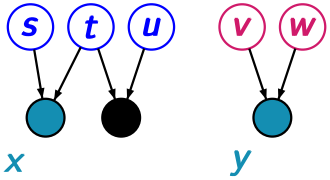


partition $\mathcal{B}$



superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow \mathit{Refine}(\mathcal{B}, C)$



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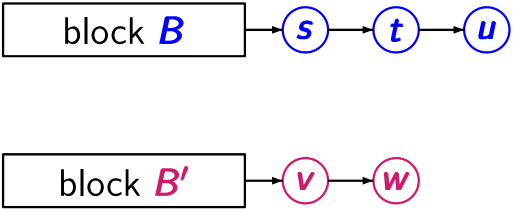

...

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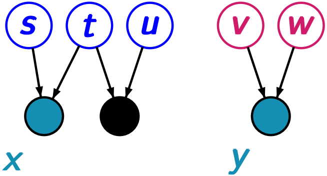

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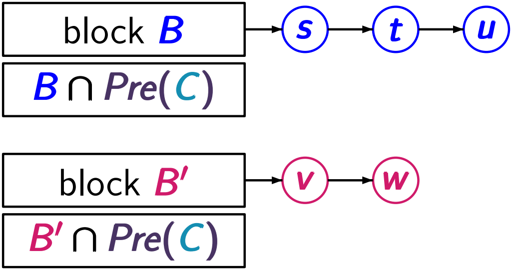

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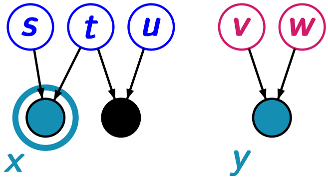

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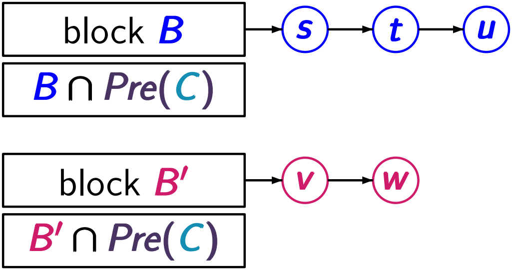

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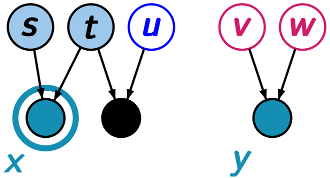
Example: refinement operator



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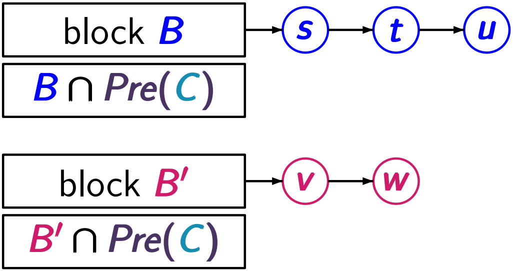
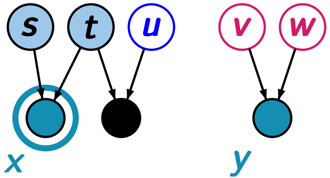


...

Example: refinement operator



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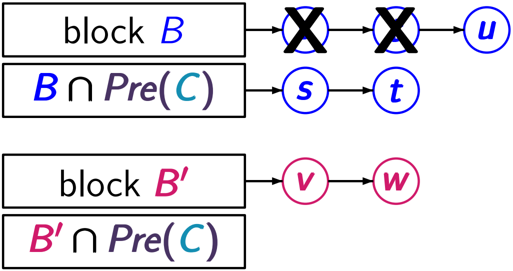
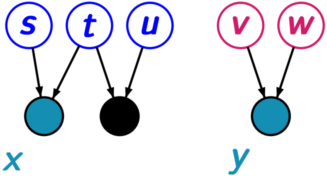


...

Example: refinement operator



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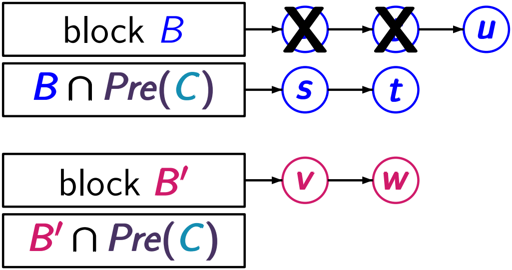
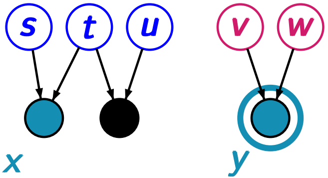


$\leftarrow$ new block $B \setminus \mathit{Pre}(C)$

...

Example: refinement operator



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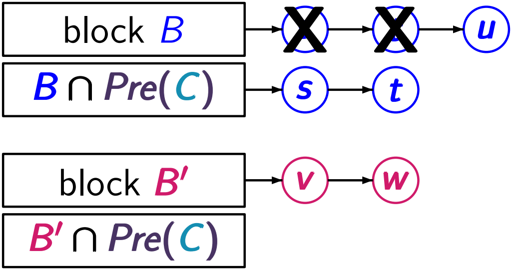

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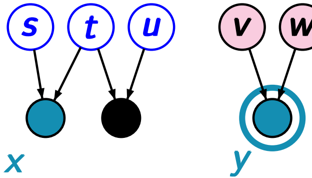
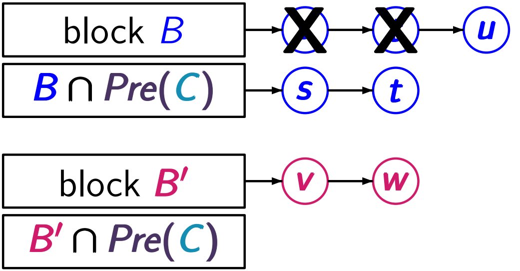


superblock $C = \{x, y\}$

partition $\mathcal{B} \rightsquigarrow Refine(\mathcal{B}, C)$



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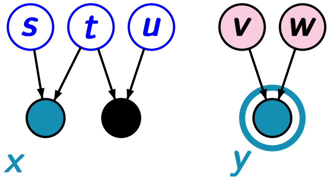

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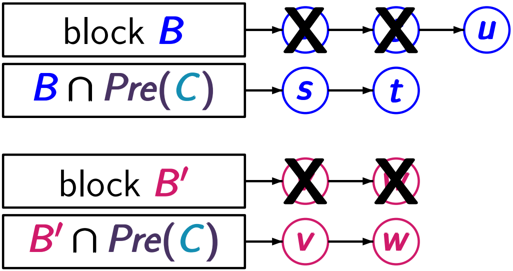

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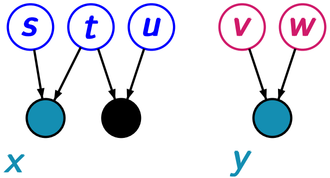

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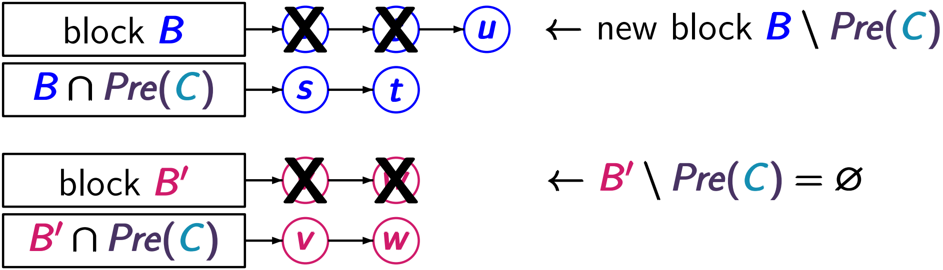

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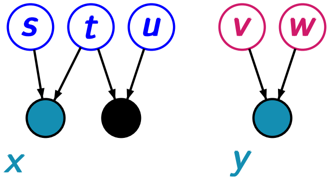

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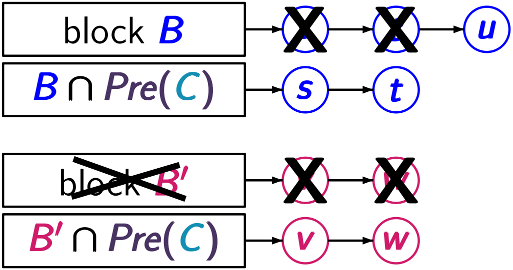
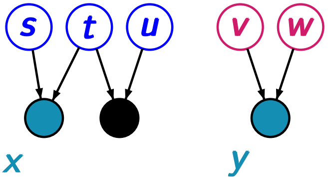


$\leftarrow$ new block $B \setminus Pre(C)$

$\leftarrow B' \setminus Pre(C) = \varnothing$

...

Example: refinement operator



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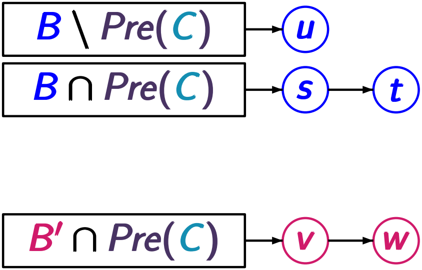
$Refine(\mathcal{B}, C)$



...

$\mathcal{B} := \mathcal{B}_{AP}$

`WHILE` there is a splitter C for $\mathcal{B}$ `DO`

select such a splitter C;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

`OD`

return $\mathcal{B}$

Partitioning splitter algorithm

$\mathcal{B} := \mathcal{B}_{AP}$ ⟵ time complexity: $\mathcal{O}(|S| \cdot |AP|)$

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WHILE there is a splitter C for B DO
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$$\mathcal{O}\Big(\sum_{C} \big(\sum_{s' \in C} |Pre(s')| + |C| \big) \; + \; |S| \cdot |AP| \Big)$$

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+ cost for splitter search and management

Partitioning splitter algorithms

2 instances of the partitioning splitter algorithm
that differ in the choice and management of splitters

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2 instances of the partitioning splitter algorithm
that differ in the choice and management of splitters

- *Kanellakis-Smolka algorithm:*

  refinement according to all blocks of the partition of the previous iteration

- *Paige-Tarjan-algorithm:*

  simultaneous refinement according to 2 superblocks

Kanellakis-Smolka algorithm

Kanellakis-Smolka algorithm

```
B := B_AP;  B_old := {S}
REPEAT
    B_old := B;
    FOR ALL  C ∈ B_old DO  B := Refine(B, C) OD
UNTIL  B = B_old
return B
```

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\text{old}} := \{S\}$

REPEAT

  $\mathcal{B}_{\text{old}} := \mathcal{B}$;

  FOR ALL $C \in \mathcal{B}_{\text{old}}$ DO $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$ OD

UNTIL $\mathcal{B} = \mathcal{B}_{\text{old}}$

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Kanellakis-Smolka algorithm

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ ⟵ cost: $\mathcal{O}(|S| \cdot |AP|)$

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Kanellakis-Smolka algorithm


```
𝓑 := 𝓑_AP;  𝓑_old := {S}      ←  cost: 𝒪(|S|·|AP|)
REPEAT
     𝓑_old := 𝓑;
     FOR ALL  C ∈ 𝓑_old DO  𝓑 := Refine(𝓑, C) OD
UNTIL  𝓑 = 𝓑_old
return 𝓑
```

- maximal $|S|$ iterations

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{\mathsf{old}} := \{S\}$ $\longleftarrow$ cost: $\mathcal{O}(|S| \cdot |AP|)$

REPEAT

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- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs

$$\mathcal{O}(|\mathit{Pre}(s')| + 1)$$

Kanellakis-Smolka algorithm

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REPEAT
    B_old := B;
    FOR ALL C ∈ B_old DO B := Refine(B, C) OD
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```

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
  $$\mathcal{O}(|Pre(s')| + 1)$$
- cost per iteration: $\mathcal{O}(m + |S|)$

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$ ⟵ cost: $\mathcal{O}(|S| \cdot |AP|)$

REPEAT

$\mathcal{B}_{old} := \mathcal{B}$;

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  if m = number of edges $= \sum_{s'} |Pre(s')|$

Kanellakis-Smolka algorithm

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- maximal $|S|$ iterations
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  if m = number of edges $= \sum_{s'} |Pre(s')| \geq |S|$

$\mathcal{B} := \mathcal{B}_{AP}$; $\mathcal{B}_{old} := \{S\}$

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return $\mathcal{B}$

time complexity:
$\mathcal{O}(|S| \cdot m + |S| \cdot |AP|)$

- maximal $|S|$ iterations
- per iteration: each state $s' \in C$ causes the costs
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- cost per iteration: $\mathcal{O}(m + |S|) = \mathcal{O}(m)$
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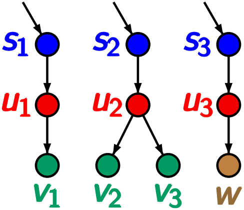
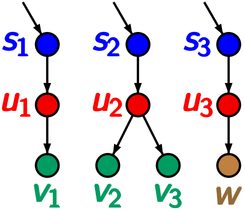
Example: Kanellakis-Smolka algorithm



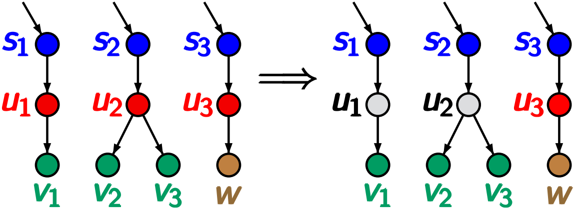
Example: Kanellakis-Smolka algorithm



1. iteration:

1. refinement w.r.t. $\{v_1, v_2, v_3\}$

Example: Kanellakis-Smolka algorithm



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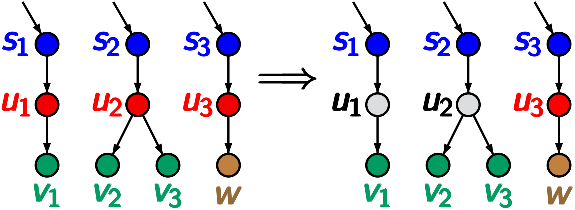

1. iteration:

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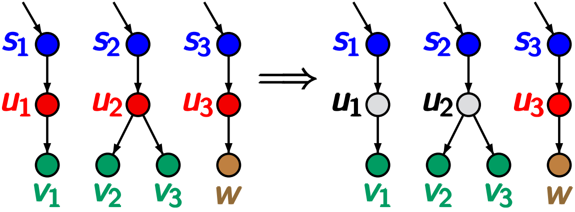
2. refinement w.r.t. $\{w\}$: no changes

Example: Kanellakis-Smolka algorithm



1. iteration:

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2. refinement w.r.t. $\{w\}$: no changes
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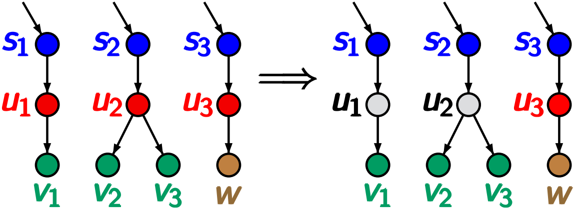
Example: Kanellakis-Smolka algorithm



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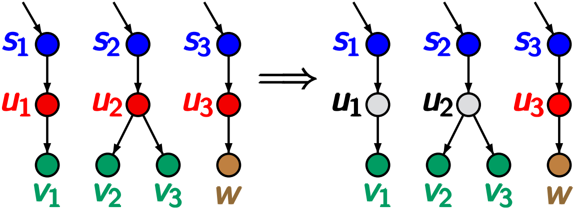
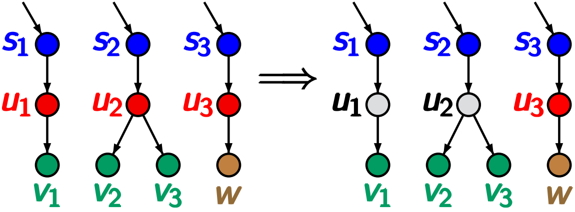
1. refinement w.r.t. $\{v_1, v_2, v_3\}$
2. refinement w.r.t. $\{w\}$: no changes
3. refinement w.r.t. $\{s_1, s_2, s_3\}$: no changes
4. refinement w.r.t. $\{u_1, u_2, u_3\}$: no changes

Example: Kanellakis-Smolka algorithm



2. iteration:

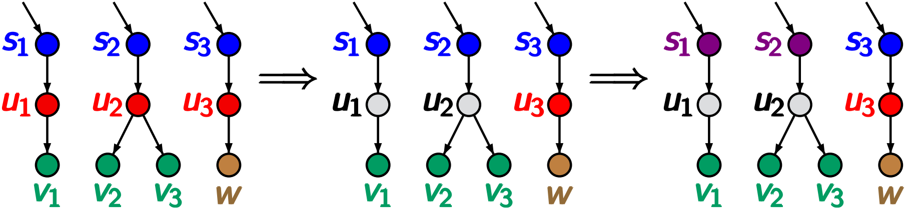
Example: Kanellakis-Smolka algorithm



2. iteration:

1. refinement w.r.t. $\{u_3\}$

Example: Kanellakis-Smolka algorithm



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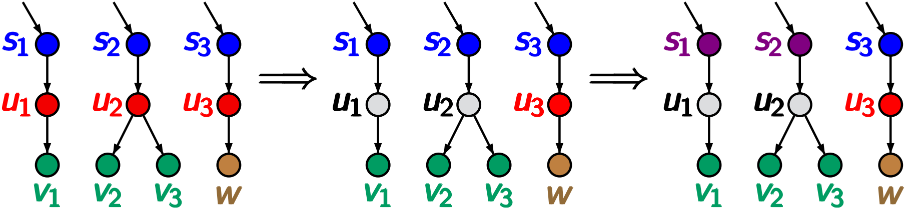
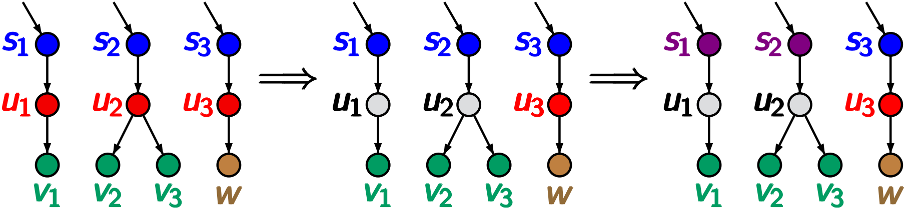
Example: Kanellakis-Smolka algorithm



2. iteration:

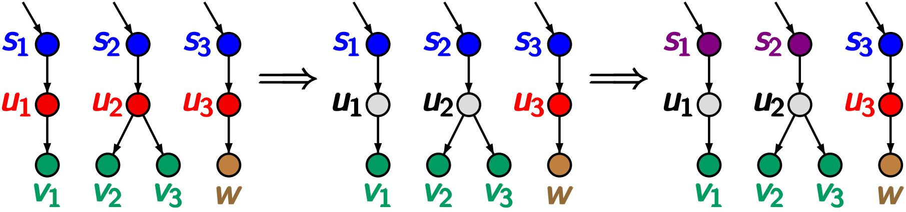
1. refinement w.r.t. $\{u_3\}$
2. refinement w.r.t. other blocks of the first iteration: no changes

Example: Kanellakis-Smolka algorithm



3. iteration:

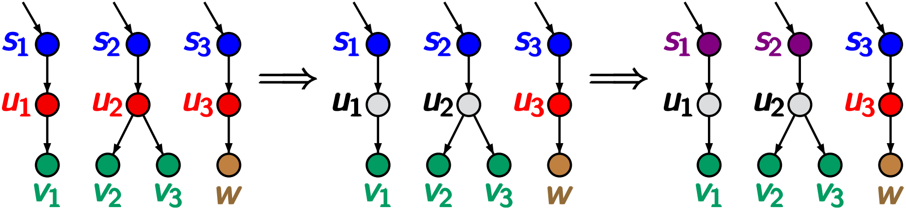
Example: Kanellakis-Smolka algorithm



3. iteration:

refinement w.r.t. all blocks of the second iteration:
no changes

Example: Kanellakis-Smolka algorithm



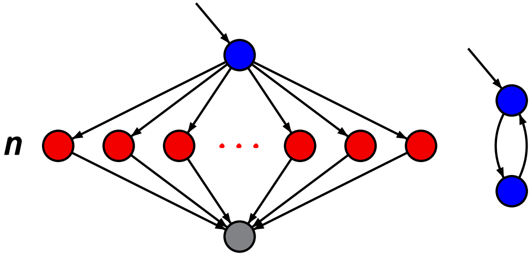
3. iteration:

refinement w.r.t. all blocks of the second iteration: no changes

6 bisimulation equivalence classes:

$\{s_1, s_2\}, \{s_3\}, \{u_1, u_2\}, \{u_3\}, \{v_1, v_2, v_3\}, \{w\}$

Partitioning splitter algorithm



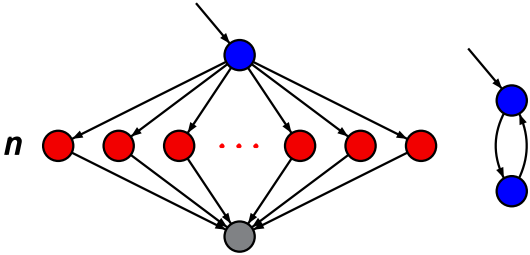
$AP = \{a, b\}$

$\widehat{=} \{a\}$

$\widehat{=} \{b\}$

$\widehat{=} \varnothing$

Partitioning splitter algorithm



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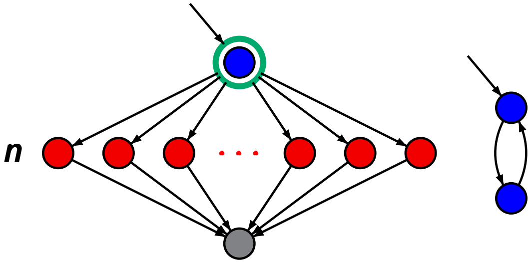
refinement w.r.t. ●:

Partitioning splitter algorithm



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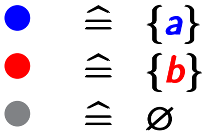
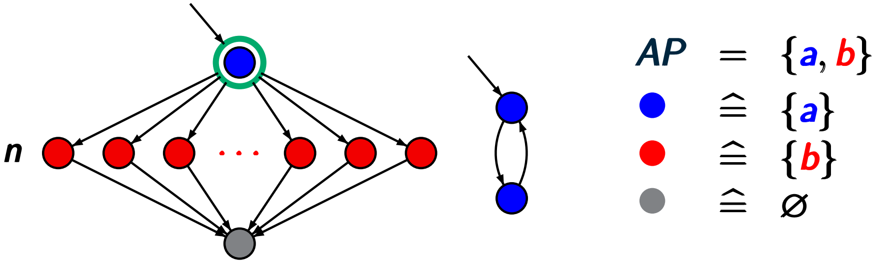


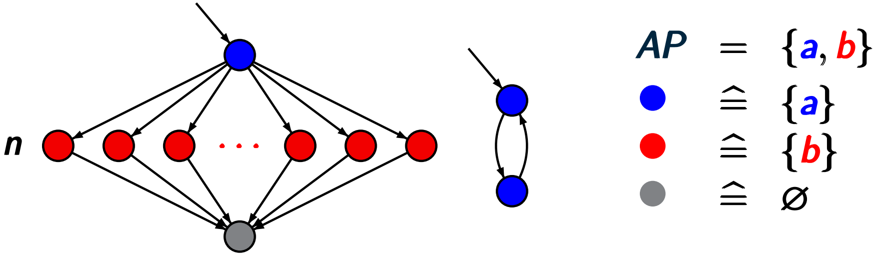
refinement w.r.t. ●:



refinement w.r.t. ●: causes the costs

$$\sum_{s'} |Pre(s')| = n$$

Partitioning splitter algorithm



refinement w.r.t. ●: causes the costs

$$\sum_{s'} \left| Pre(s') \right| = n$$

alternatively: refinement w.r.t. ●: constant costs

Kanellakis-Smolka algorithm:

initially: $\mathcal{B}_{\text{old}} = \mathcal{B} = \mathcal{B}_{AP}$

Partitioning splitter algorithms

Kanellakis-Smolka algorithm:

initially: $\mathcal{B}_{\mathsf{old}} = \mathcal{B} = \mathcal{B}_{AP}$

iteration: stabilization for each block in $\mathcal{B}_{\mathsf{old}}$

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loop invariant: $\mathcal{B}$ finer than $\mathcal{B}_{\text{old}}$ and coarser than $S/\!\sim$

Partitioning splitter algorithms

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Paige-Tarjan algorithm:

Partitioning splitter algorithms

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Paige-Tarjan algorithm:

loop invariant:

(1) $\mathcal{B}$ finer than $\mathcal{B}_{\text{old}}$ and coarser than $S/\sim$

(2) $\mathcal{B}$ is stable for each block in $\mathcal{B}_{\text{old}}$

Partitioning splitter algorithms

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Partitioning splitter algorithms

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iteration: ternary refinement operator

initially: $\mathcal{B}_{\text{old}} = \{S\}$

Partitioning splitter algorithms

Kanellakis-Smolka algorithm:

initially: $\mathcal{B}_{\text{old}} = \mathcal{B} = \mathcal{B}_{AP}$

iteration: stabilization for each block in $\mathcal{B}_{\text{old}}$

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loop invariant:

(1) $\mathcal{B}$ finer than $\mathcal{B}_{\text{old}}$ and coarser than $S/\sim$
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iteration: ternary refinement operator

initially: $\mathcal{B}_{\text{old}} = \{S\}$, $\mathcal{B} = \mathit{Refine}(\mathcal{B}_{AP}, S)$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S

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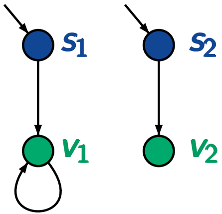
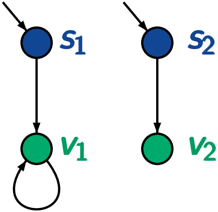


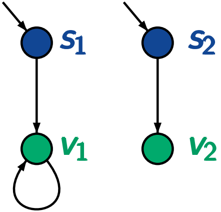
$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \Big\{ \{s_1, s_2\}, \{v_1, v_2\} \Big\}$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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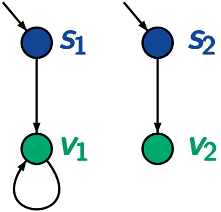
$Pre(S)$ = set of nonterminal states

$= \{s_1, s_2, v_1\}$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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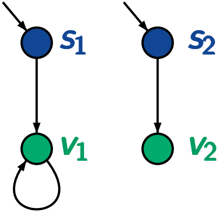
$Pre(S)$ = set of nonterminal states $= \{s_1, s_2, v_1\}$

$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$

$$\{v_1, v_2\} \setminus Pre(S) = \{v_2\}$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



state space $S = \{s_1, s_2, v_1, v_2\}$

$$\mathcal{B}_{AP} = \{ \{s_1, s_2\}, \{v_1, v_2\} \}$$

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$$\{v_1, v_2\} \cap Pre(S) = \{v_1\}$$
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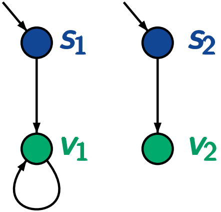
initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S)$$

$\mathcal{B}_{AP}$ is generally not stable w.r.t. S



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initial partition of Paige/Tarjan algorithm:

$$Refine(\mathcal{B}_{AP}, S) = \Big\{ \{s_1, s_2\}, \{v_1\}, \{v_2\} \Big\}$$

Paige-Tarjan algorithm

Paige-Tarjan algorithm

$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

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  select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

OD

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```
WHILE B ≠ B_old DO
```

WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

  select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

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OD

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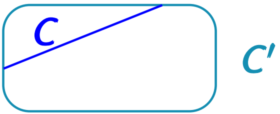

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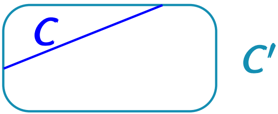

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select a block $C' \in \mathcal{B}_{old} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

refine $\mathcal{B}$
w.r.t. C and $C' \setminus C$

OD

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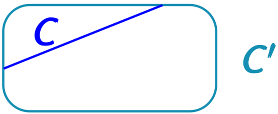

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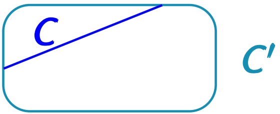
Paige-Tarjan algorithm



$\mathcal{B}_{\mathsf{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

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  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C)$

  $\mathcal{B} := \mathit{Refine}(\mathcal{B}, C')$

> refine $\mathcal{B}$ simultaneously
> w.r.t. C and $C' \setminus C$

OD

$\mathcal{B}_{old} := \{S\}$; $\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$;

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select a block $C' \in \mathcal{B}_{old} \setminus \mathcal{B}$;

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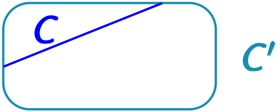

$\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

OD

Paige-Tarjan algorithm

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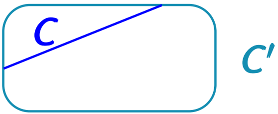


```
WHILE B ≠ B_old DO
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$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

add C and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$

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OD
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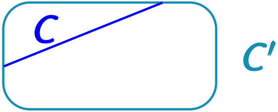
$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$;



WHILE $\mathcal{B} \neq \mathcal{B}_{\text{old}}$ DO

select a block $C' \in \mathcal{B}_{\text{old}} \setminus \mathcal{B}$;

select a block $C \in \mathcal{B}$ with $C \subseteq C'$ and $|C| \leq |C'|/2$;

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

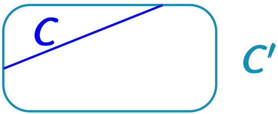
refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

add C and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$ and remove C' from $\mathcal{B}_{\text{old}}$

OD

Paige-Tarjan algorithm

$\mathcal{B}_{\text{old}} := \{S\}$; $\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$;



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$\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

refine $\mathcal{B}$ simultaneously w.r.t. C and $C' \setminus C$

add C and $C' \setminus C$ to $\mathcal{B}_{\text{old}}$ and remove C' from $\mathcal{B}_{\text{old}}$

OD

loop invariant: $\mathcal{B}$ is stable w.r.t. each block in $\mathcal{B}_{\text{old}}$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'

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Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
- C a block in $\mathcal{B}$ s.t. $C \subseteq C'$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
- C a block in $\mathcal{B}$ s.t. $C \subseteq C'$

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
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simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

The ternary refinement operator

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
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simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
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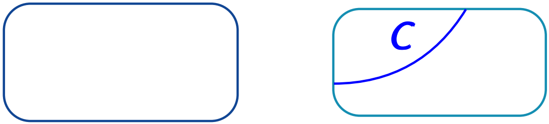

block B

superblock C'

Let $\mathcal{B}$ be a partition and

- C' a superblock of $\mathcal{B}$ s.t. $\mathcal{B}$ is stable w.r.t. C'
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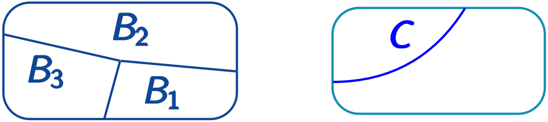


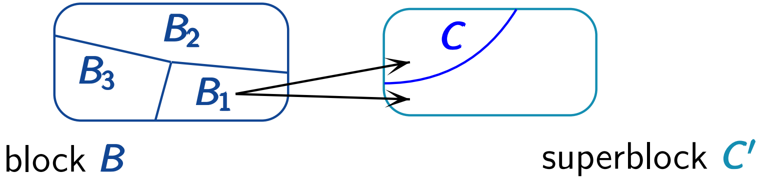
block B superblock C'

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$

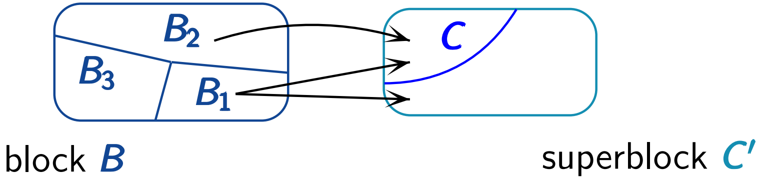
The ternary refinement operator

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$
$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

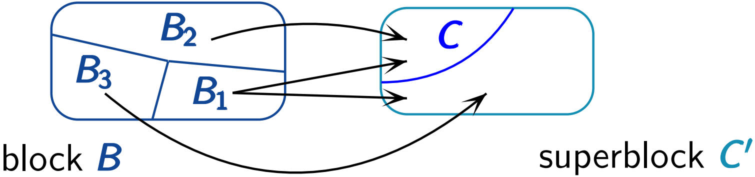
The ternary refinement operator

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



$$\begin{aligned} B_1 &= B \cap Pre(C) \cap Pre(C' \setminus C) \\ B_2 &= (B \cap Pre(C)) \setminus Pre(C' \setminus C) \\ B_3 &= (B \cap Pre(C' \setminus C)) \setminus Pre(C) \end{aligned}$$

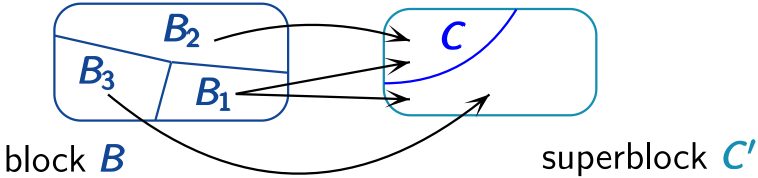
The ternary refinement operator

simultaneous refinement of $\mathcal{B}$ w.r.t. C and $C' \setminus C$:

$$Refine(\mathcal{B}, C, C'\setminus C) = \bigcup_{B\in\mathcal{B}} Refine(B, C, C'\setminus C)$$

where for block $B \subseteq Pre(C')$:

$$Refine(B, C, C'\setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$



for block B with $B \cap Pre(C') = \varnothing$:

$$Refine(B, C, C'\setminus C) = \{B\}$$

Stability of $Refine(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq Pre(C') \quad \text{or} \quad B \cap Pre(C') = \varnothing$$

Stability of $Refine(\mathcal{B}, C, C' \setminus C)$

Suppose that for all blocks $B \in \mathcal{B}$:

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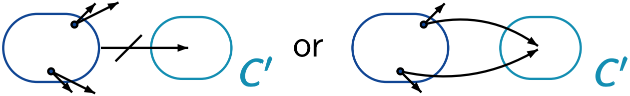
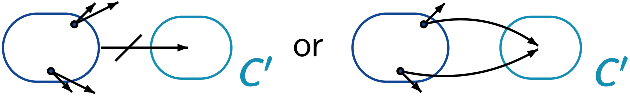
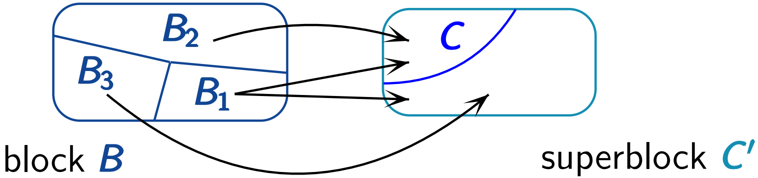


Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq Pre(C') \quad \text{or} \quad B \cap Pre(C') = \varnothing$$

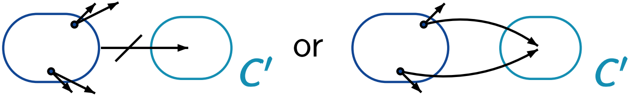


Then the new blocks B_1, B_2, B_3 in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.



Suppose that for all blocks $B \in \mathcal{B}$:

$$B \subseteq Pre(C') \quad \text{or} \quad B \cap Pre(C') = \varnothing$$

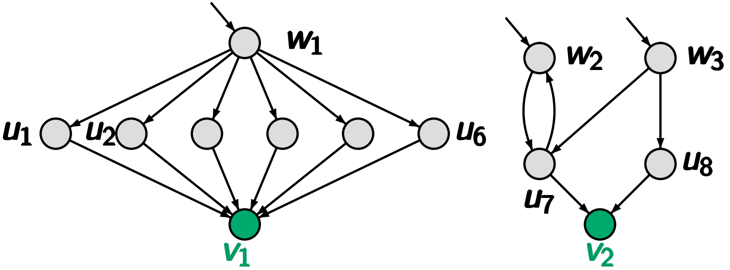


Then the new blocks B_1, B_2, B_3 in $Refine(B, C, C' \setminus C)$ are stable w.r.t. the superblocks C and $C' \setminus C$.

If $\mathcal{B}$ is stable w.r.t. all blocks in $\mathcal{B}_{old}$ and $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subsetneq C'$ then $Refine(\mathcal{B}, C, C' \setminus C)$ is stable w.r.t. all blocks in the partition

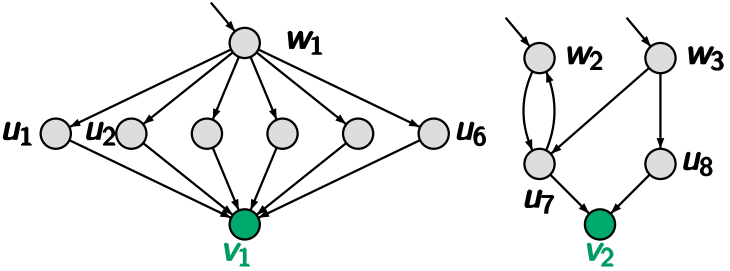
$$(\mathcal{B}_{old} \setminus \{C'\}) \cup \{C, C' \setminus C\}$$

Example: Paige-Tarjan algorithm



$AP = \{green, gray\}, \quad \mathcal{B}_{old} = \{S\}$

Example: Paige-Tarjan algorithm

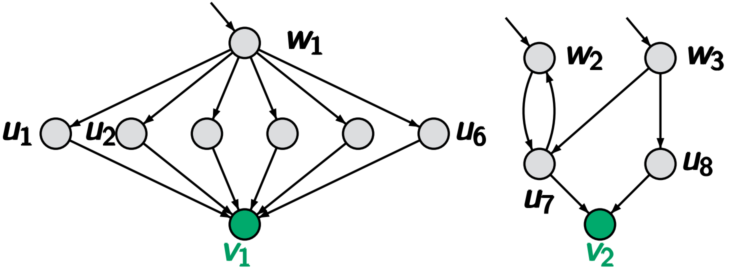


$AP = \{green, gray\}$, $\mathcal{B}_{old} = \{S\}$

initial partition:

$$\mathcal{B}_0 = Refine(\mathcal{B}_{AP}, S) = \mathcal{B}_{AP}$$
$$= \{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\}$$

Example: Paige-Tarjan algorithm



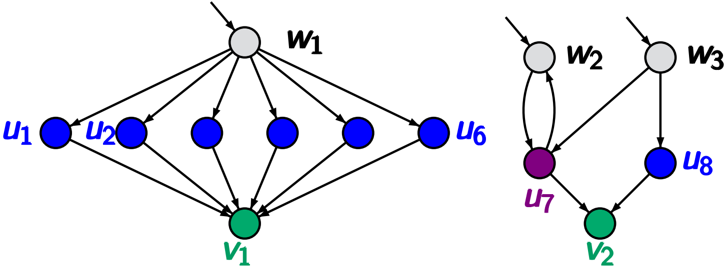
initially: $\mathcal{B}_{old} = \{S\}$

$\mathcal{B}_0 = \{\{v_1, v_2\}, \{u_1, \ldots, u_8, w_1, w_2, w_3\}\}$

first refinement step:

$Refine(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\})$

Example: Paige-Tarjan algorithm



initially: $\mathcal{B}_{old} = \{S\}$

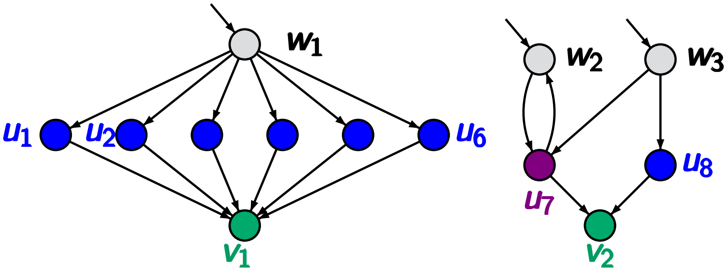
$\mathcal{B}_0 = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\big\}$

first refinement step:

$$\mathit{Refine}(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\}) =$$

$$\mathcal{B}_1 = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\big\}$$

Example: Paige-Tarjan algorithm



initially: $\mathcal{B}_{\text{old}} = \{S\}$

$\mathcal{B}_0 = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\big\}$

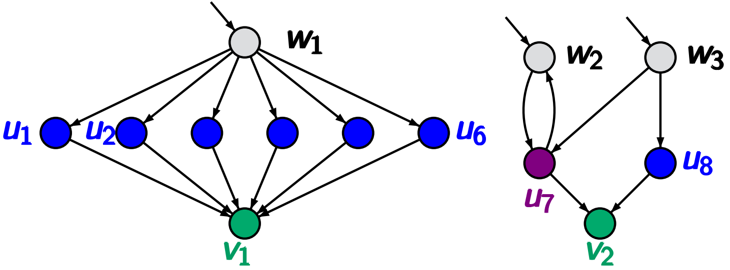
first refinement step:

$Refine(\mathcal{B}_0, \{v_1, v_2\}, S \setminus \{v_1, v_2\}) =$

$\mathcal{B}_1 = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\big\}$

$\mathcal{B}_{\text{old}} = \big\{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\big\}$

Example: Paige-Tarjan algorithm



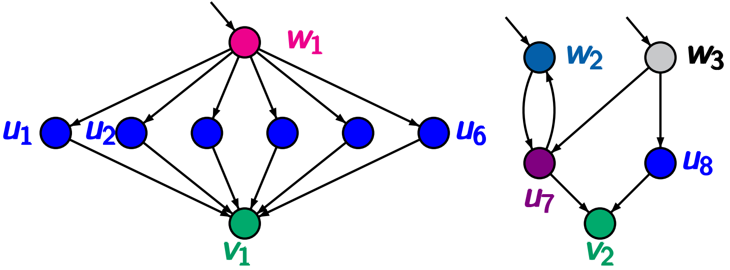
first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{old} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$Refine(\mathcal{B}_1, ?, ?)$

Example: Paige-Tarjan algorithm



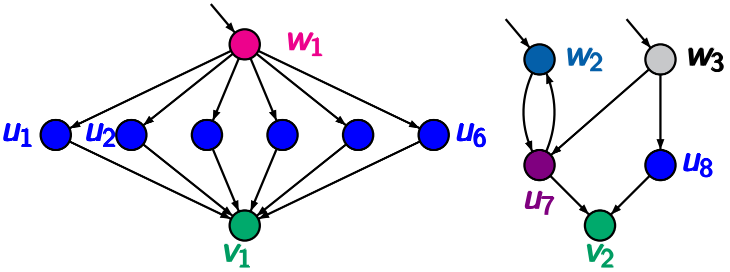
first refinement step:

$$\mathcal{B}_1 = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1, w_2, w_3\}\}$$
$$\mathcal{B}_{old} = \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8, u_7, w_1, w_2, w_3\}\}$$

second refinement step:

$$Refine(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$

Example: Paige-Tarjan algorithm



first refinement step:

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second refinement step:

$$Refine(\mathcal{B}_1, \{u_7\}, \{u_1, \ldots, u_6, u_8, w_1, w_2, w_3\})$$
$$= \{\{v_1, v_2\}, \{u_1, \ldots, u_6, u_8\}, \{u_7\}, \{w_1\}, \{w_2\}, \{w_3\}\}$$

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{\mathsf{old}} := \{S\}$;

```
WHILE B ≠ B_old DO
    select C' ∈ B_old, C ∈ B s.t. C ⊆ C', |C| ≤ |C'|/2;
    add C and C' \ C to B_old and remove C' from B_old
    B := Refine(B, C, C' \ C)
OD
return B
```

WHILE $\mathcal{B} \neq \mathcal{B}_{\mathsf{old}}$ DO

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and remove C' from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

return $\mathcal{B}$

```
B := Refine(B_AP, S);  B_old := {S};
WHILE B ≠ B_old DO
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$\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

OD

return $\mathcal{B}$

efficient implementation of $Refine(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}(|C| + |Pre(C)|)$

$\mathcal{B} := Refine(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

  select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

  add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

  $\mathcal{B} := Refine(\mathcal{B}, C, C' \setminus C)$

OD

return $\mathcal{B}$

efficient implementation of $Refine(\mathcal{B}, C, \ldots)$ with time complexity $\mathcal{O}\big(|C| + |Pre(C)|\big)$ uses counters

$$\delta(s, D) = |Post(s) \cap D| \text{ for } D \in \mathcal{B}_{old}$$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$\nwarrow$ $D \in \mathcal{B}_{\text{old}}$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\nearrow$ $\quad$ $\nwarrow$ $D \in \mathcal{B}_{\mathsf{old}}$

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

using counters $\delta(s, D) = |Post(s) \cap D|$

$s \in Pre(D)$ $\quad$ $D \in \mathcal{B}_{\text{old}}$

step 1: compute $\delta(\ldots)$ for the new blocks C and $C' \setminus C$ in $\mathcal{B}_{\text{old}}$

Details of the Paige-Tarjan algorithm

implementation of

$$Refine(\mathcal{B}, C, C' \setminus C) = \bigcup_{B \in \mathcal{B}} Refine(B, C, C' \setminus C)$$

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$s \in Pre(D)$ $\quad$ $D \in \mathcal{B}_{\text{old}}$

step 1: compute $\delta(\ldots)$ for the new blocks C and $C' \setminus C$ in $\mathcal{B}_{\text{old}}$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ $\boxed{\text{for } s \in Pre(C')}$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ $\boxed{\text{for } s \in \mathit{Pre}(C')}$

$$\delta(s, C) = |\mathit{Post}(s) \cap C|$$

$$\delta(s, C' \setminus C) = |\mathit{Post}(s) \cap (C' \setminus C)|$$

step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ for $s \in \mathit{Pre}(C')$

$$\delta(s, C) = |\mathit{Post}(s) \cap C|$$
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step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \cap \mathit{Pre}(C') = \varnothing$ we have:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\leftarrow$ for $s \in Pre(C')$

$$\delta(s, C) = |Post(s) \cap C|$$

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for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

Implementation of $\mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

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step 2: compute $\mathit{Refine}(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq \mathit{Pre}(C')$:

$$\mathit{Refine}(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ for $s \in Pre(C')$

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$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = B \cap Pre(C) \cap Pre(C' \setminus C)$$
$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$
$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ for $s \in Pre(C')$

$$\delta(s, C) = |Post(s) \cap C|$$

$$\delta(s, C' \setminus C) = |Post(s) \cap (C' \setminus C)|$$

step 2: compute $Refine(B, C, C' \setminus C)$ for all $B \in \mathcal{B}$

for $B \in \mathcal{B}$ with $B \subseteq Pre(C')$:

$$Refine(B, C, C' \setminus C) = \{B_1, B_2, B_3\} \setminus \{\varnothing\}$$

$$B_1 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) > 0\}$$

$$B_2 = (B \cap Pre(C)) \setminus Pre(C' \setminus C)$$

$$B_3 = (B \cap Pre(C' \setminus C)) \setminus Pre(C)$$

Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ for $s \in Pre(C')$

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Implementation of $Refine(\mathcal{B}, C, C' \setminus C)$

step 1: compute $\delta(s, C)$, $\delta(s, C' \setminus C)$ $\longleftarrow$ for $s \in Pre(C')$

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$$B_2 = \{s \in B : \delta(s, C) > 0, \delta(s, C' \setminus C) = 0\}$$

$$B_3 = \{s \in B : \delta(s, C) = 0, \delta(s, C' \setminus C) > 0\}$$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

```
WHILE  B ≠ B_old DO
```

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;
add C and $C' \backslash C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

```
OD
```

Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\mathcal{B}_{old} := \{S\}$;

FOR ALL $s \in S$ DO $\delta(s, S) := |\mathit{Post}(s)|$ OD

WHILE $\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t. $C \subseteq C'$, $|C| \leq |C'|/2$;

add C and $C' \backslash C$ to $\mathcal{B}_{old}$ and remove C' from $\mathcal{B}_{old}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

```
B := Refine(B_AP, S); B_old := {S};
FOR ALL s ∈ S DO  δ(s, S) := |Post(s)| OD
WHILE B ≠ B_old DO
  select C' ∈ B_old, C ∈ B s.t. C ⊆ C', |C| ≤ |C'|/2;
  add C and C'\C to B_old and remove C' from B_old
    FOR ALL s ∈ Pre(C) DO  δ(s, C) := 0 OD
    FOR ALL s' ∈ C DO
     FOR ALL s ∈ Pre(s') DO  δ(s, C) := δ(s, C)+1 OD
    OD


  B := Refine(B, C, C' \ C)
OD
```

```
B := Refine(B_AP, S); B_old := {S};
FOR ALL s ∈ S DO δ(s,S) := |Post(s)| OD
WHILE B ≠ B_old DO
  select C' ∈ B_old, C ∈ B s.t. C ⊆ C', |C| ≤ |C'|/2;
  add C and C'\C to B_old and remove C' from B_old
    FOR ALL s ∈ Pre(C) DO δ(s,C) := 0 OD
    FOR ALL s' ∈ C DO
      FOR ALL s ∈ Pre(s') DO δ(s,C) := δ(s,C)+1 OD
    OD
    FOR ALL s ∈ Pre(C) DO
          δ(s,C'\C) := δ(s,C')−δ(s,C) OD
  B := Refine(B, C, C' \ C)
OD
```

Complexity of the Paige-Tarjan algorithm PARTSPLITALG5.3-26A

Complexity of the Paige-Tarjan algorithm

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

n = # states $= |S|$

m = # transitions

let $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ be a finite TS

$$n = \#\text{ states} = |S|$$

$$m = \#\text{ transitions} = \sum_{s \in S} |Pre(s)|$$

let $\mathcal{T} = (S, Act, \to, S_0, AP, L)$ be a finite TS

$$n = \#\text{ states} = |S|$$

$$m = \#\text{ transitions} = \sum_{s \in S} |Pre(s)|$$

in what follows, we suppose $m \geq n$

Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{\mathsf{AP}}, S)$;

$\mathcal{B}_{\mathsf{old}} := \{S\}$;

```
WHILE B ≠ B_old DO
```

select $C' \in \mathcal{B}_{\mathsf{old}}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{\mathsf{old}}$ and
remove C' from $\mathcal{B}_{\mathsf{old}}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

```
OD
```

Complexity of the Paige-Tarjan algorithm

$\mathcal{B} := \mathit{Refine}(\mathcal{B}_{AP}, S)$; $\longleftarrow$ complexity: $\mathcal{O}(n \cdot |AP|)$

$\mathcal{B}_{old} := \{S\}$;

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WHILE
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$\mathcal{B} \neq \mathcal{B}_{old}$ DO

select $C' \in \mathcal{B}_{old}$, $C \in \mathcal{B}$ s.t.
$C \subseteq C'$ and $|C| \leq |C'|/2$;

add C and $C' \setminus C$ to $\mathcal{B}_{old}$ and
remove C' from $\mathcal{B}_{old}$

$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$

OD

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$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$ ⟵ time complexity: $\sum_{s' \in C} |\mathit{Pre}(s')| + 1$

OD

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$\mathcal{B} := \mathit{Refine}(\mathcal{B}, C, C' \setminus C)$ ⟵ time complexity: $\mathcal{O}(|C| + |\mathit{Pre}(C)|)$

OD

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OD

total cost for all refinement operations: $\mathcal{O}(m \cdot \log n)$