Semantics and Verification 2007

Lecture 7

- bisimulation as a fixed point
- Hennessy-Milner logic with recursively defined variables
- game semantics and temporal properties of reactive systems
- characteristic property

Tarski's Fixed Point Theorem – Summary

Let $(D, \sqsubseteq)$ be a complete lattice and let $f : D \to D$ be a monotonic function.

Tarski's Fixed Point Theorem

Then f has a unique largest fixed point z_{max} and a unique least fixed point z_{min} given by:

$$z_{max} \stackrel{\text{def}}{=} \sqcup\{x \in D \mid x \sqsubseteq f(x)\}$$

$$z_{min} \stackrel{\text{def}}{=} \sqcap\{x \in D \mid f(x) \sqsubseteq x\}$$

Computing Fixed Points in Finite Lattices

If D is a finite set then there exist integers $M, m > 0$ such that

- $z_{max} = f^M(\top)$
- $z_{min} = f^m(\bot)$

Definition of Strong Bisimulation

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS.

Strong Bisimulation

A binary relation $R \subseteq Proc \times Proc$ is a strong bisimulation iff whenever $(s, t) \in R$ then for each $a \in Act$:

- if $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ for some t' such that $(s', t') \in R$
- if $t \xrightarrow{a} t'$ then $s \xrightarrow{a} s'$ for some s' such that $(s', t') \in R$.

Two processes $p, q \in Proc$ are strongly bisimilar $(p \sim q)$ iff there exists a strong bisimulation R such that $(p, q) \in R$.

$$\sim \; = \bigcup \{R \mid R \text{ is a strong bisimulation}\}$$

Strong Bisimulation as a Greatest Fixed Point

Function $\mathcal{F} : 2^{(Proc \times Proc)} \longrightarrow 2^{(Proc \times Proc)}$

Let $S \subseteq Proc \times Proc$. Then we define $\mathcal{F}(S)$ as follows:

$(s, t) \in \mathcal{F}(S)$ if and only if for each $a \in Act$:

- if $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ for some t' such that $(s', t') \in S$
- if $t \xrightarrow{a} t'$ then $s \xrightarrow{a} s'$ for some s' such that $(s', t') \in S$.

Observations

- $(2^{(Proc \times Proc)}, \subseteq)$ is a complete lattice and $\mathcal{F}$ is monotonic
- S is a strong bisimulation if and only if $S \subseteq \mathcal{F}(S)$

Strong Bisimilarity is the Greatest Fixed Point of $\mathcal{F}$

$$\sim = \bigcup \{ S \in 2^{(Proc \times Proc)} \mid S \subseteq \mathcal{F}(S) \}$$

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