HML with One Recursively Defined Variable

Syntax of Formulae

Formulae are given by the following abstract syntax

$$F ::= X \mid tt \mid ff \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \langle a \rangle F \mid [a]F$$

where $a \in Act$ and X is a distinguished variable with a definition

- $X \stackrel{\min}{=} F_X$, or $X \stackrel{\max}{=} F_X$

such that F_X is a formula of the logic (can contain X).

How to Define Semantics?

For every formula F we define a function $O_F : 2^{Proc} \to 2^{Proc}$ s.t.

- if S is the set of processes that satisfy X then
- $O_F(S)$ is the set of processes that satisfy F.

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Definition of $O_F : 2^{Proc} \longrightarrow 2^{Proc}$ (let $S \subseteq Proc$)

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\begin{aligned}
O_X(S) &= S \\
O_{tt}(S) &= Proc \\
O_{ff}(S) &= \emptyset \\
O_{F_1 \wedge F_2}(S) &= O_{F_1}(S) \cap O_{F_2}(S) \\
O_{F_1 \vee F_2}(S) &= O_{F_1}(S) \cup O_{F_2}(S) \\
O_{\langle a \rangle F}(S) &= \langle \cdot a \cdot \rangle O_F(S) \\
O_{[a] F}(S) &= [\cdot a \cdot] O_F(S)
\end{aligned}
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O_F is monotonic for every formula F

$$S_1 \subseteq S_2 \Rightarrow O_F(S_1) \subseteq O_F(S_2)$$

Proof: easy (structural induction on the structure of F).

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Semantics

Observation

We know that $(2^{Proc}, \subseteq)$ is a complete lattice and O_F is monotonic, so O_F has a unique greatest and least fixed point.

Semantics of the Variable X

- If $X \stackrel{\max}{=} F_X$ then
$$[\![X]\!] = \bigcup \{S \subseteq Proc \mid S \subseteq O_{F_X}(S)\}.$$
- If $X \stackrel{\min}{=} F_X$ then
$$[\![X]\!] = \bigcap \{S \subseteq Proc \mid O_{F_X}(S) \subseteq S\}.$$

Game Characterization

Intuition: the attacker claims $s \not\models F$, the defender claims $s \models F$.

Configurations of the game are of the form (s, F)

- (s, tt) and (s, ff) have no successors
- (s, X) has one successor (s, F_X)
- $(s, F_1 \wedge F_2)$ has two successors (s, F_1) and (s, F_2) (selected by the attacker)
- $(s, F_1 \vee F_2)$ has two successors (s, F_1) and (s, F_2) (selected by the defender)
- $(s, [a]F)$ has successors (s', F) for every s' s.t. $s \xrightarrow{a} s'$ (selected by the attacker)
- $(s, \langle a \rangle F)$ has successors (s', F) for every s' s.t. $s \xrightarrow{a} s'$ (selected by the defender)

Who is the Winner?

Play is a maximal sequence of configurations formed according to the rules given on the previous slide.

Finite Play

- The attacker is the winner of a finite play if the defender gets stuck or the players reach a configuration (s, ff).
- The defender is the winner of a finite play if the attacker gets stuck or the players reach a configuration (s, tt).

Infinite Play

- The attacker is the winner of an infinite play if X is defined as $X \stackrel{\min}{=} F_X$.
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Game Characterization

Theorem

- $s \models F$ if and only if the defender has a universal winning strategy from (s, F)
- $s \not\models F$ if and only if the attacker has a universal winning strategy from (s, F)

Selection of Temporal Properties

- *Inv(F)*: $X \overset{\max}{=} F \wedge [Act]X$
- *Pos(F)*: $X \overset{\min}{=} F \vee \langle Act\rangle X$

- *Safe(F)*: $X \overset{\max}{=} F \wedge ([Act]ff \vee \langle Act\rangle X)$
- *Even(F)*: $X \overset{\min}{=} F \vee (\langle Act\rangle tt \wedge [Act]X)$

- $F\ \mathcal{U}^w\ G$: $X \overset{\max}{=} G \vee (F \wedge [Act]X)$
- $F\ \mathcal{U}^s\ G$: $X \overset{\min}{=} G \vee (F \wedge \langle Act\rangle tt \wedge [Act]X)$

Using until we can express e.g. *Inv(F)* and *Even(F)*:

$$Inv(F) \equiv F\ \mathcal{U}^w\ ff \qquad\qquad Even(F) \equiv tt\ \mathcal{U}^s\ F$$

Selection of Temporal Properties

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Examples of More Advanced Recursive Formulae

Nested Definitions of Recursive Variables

$$X \stackrel{\min}{=} Y \vee \langle Act \rangle X \qquad Y \stackrel{\max}{=} \langle a \rangle tt \wedge \langle Act \rangle Y$$

Solution: compute first $[\![Y]\!]$ and then $[\![X]\!]$.

Mutually Recursive Definitions

$$X \stackrel{\max}{=} [a]Y \qquad Y \stackrel{\max}{=} \langle a \rangle X$$

Solution: consider a complete lattice $(2^{Proc} \times 2^{Proc}, \sqsubseteq)$ where $(S_1, S_2) \sqsubseteq (S_1', S_2')$ iff $S_1 \subseteq S_1'$ and $S_2 \subseteq S_2'$.

Theorem (Characteristic Property for Finite-State Processes)

Let s be a process with finitely many reachable states. There exists a property X_s s.t. for all processes t: $s \sim t$ if and only if $t \in [\![X_s]\!]$.

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