Real-time and Probabilistic Systems Verification

Luca Tesei

MSc in Computer Science, University of Camerino

Topics

- TCTL
- Model Checking for TCTL

More:

The slides in the following pages are taken from the material of the course “Advanced Model Checking” held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

Timed CTL

Syntax of TCTL *state-formulas* over *AP* and set C:

$$\Phi ::= \text{true} \mid a \mid g \mid \Phi \wedge \Phi \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

where $a \in AP$, $g \in ACC(C)$ and φ is a *path-formula* defined by:

$$\varphi ::= \Phi \, \mathsf{U}^{J} \, \Phi$$

where $J \subseteq \mathbb{R}_{\geqslant 0}$ is an interval whose bounds are naturals

abbreviate $[c, \infty)$ by $x > c$, $(c_1, c_2]$ by $c_1 < x \leqslant c_2$ etc.

Some abbreviations

"Always" is obtained in the following way:

$$\exists\Box^J \Phi \;=\; \neg\forall\Diamond^J \neg\Phi \quad \text{and} \quad \forall\Box^J \Phi \;=\; \neg\exists\Diamond^J \neg\Phi$$

$\exists\Box^J \Phi$ asserts that for some path during the interval J, Φ holds

$\forall\Box^J \Phi$ requires this to hold for all paths

Standard $\Box$ and $\Diamond$-operator are obtained as follows:

$$\Diamond \Phi = \Diamond^{[0,\infty)} \Phi \quad \text{and} \quad \Box \Phi = \Box^{[0,\infty)} \Phi$$

Semantics of TCTL

For state $s = \langle \ell, \eta \rangle$ in *TS*(*TA*) the satisfaction relation $\models$ is defined by:

$$
\begin{array}{lll}
s \models \text{true} & & \\
s \models a & \text{iff} & a \in L(\ell) \\
s \models g & \text{iff} & \eta \models g \\
s \models \neg \Phi & \text{iff} & \text{not } s \models \Phi \\
s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \\
s \models \exists \varphi & \text{iff} & \pi \models \varphi \text{ for some } \pi \in \mathit{Paths}_{\mathit{div}}(s) \\
s \models \forall \varphi & \text{iff} & \pi \models \varphi \text{ for all } \pi \in \mathit{Paths}_{\mathit{div}}(s)
\end{array}
$$

path quantification over time-divergent paths only

The $\Longrightarrow$ relation

For infinite path fragments in *TS*(*TA*) performing ∞ many actions let:

$$s_0 \overset{d_0}{\Longrightarrow} s_1 \overset{d_1}{\Longrightarrow} s_2 \overset{d_2}{\Longrightarrow} \ldots \qquad \text{with } d_0, d_1, d_2 \ldots \geqslant 0$$

denote the equivalence class containing all infinite path fragments induced by execution fragments of the form:

$$s_0 \underbrace{\overset{d_0^1}{\rightarrow} \ldots \overset{d_0^{k_0}}{\rightarrow}}_{\substack{\text{time passage of} \\ d_0 \text{ time-units}}} s_0{+}d_0 \overset{\alpha_1}{\longrightarrow} s_1 \underbrace{\overset{d_1^1}{\rightarrow} \ldots \overset{d_1^{k_1}}{\rightarrow}}_{\substack{\text{time passage of} \\ d_1 \text{ time-units}}} s_1{+}d_1 \overset{\alpha_2}{\longrightarrow} s_2 \underbrace{\overset{d_2^1}{\rightarrow} \ldots \overset{d_2^{k_2}}{\rightarrow}}_{\substack{\text{time passage of} \\ d_2 \text{ time-units}}} s_2{+}d_2 \overset{\alpha_3}{\longrightarrow} \ldots$$

where $k_i \in \mathbb{N}$, $d_i \in \mathbb{R}_{\geqslant 0}$ and $\alpha_i \in \mathit{Act}$ such that $\sum_{j=1}^{k_i} d_i^j = d_i$.

For $\pi \in s_0 \overset{d_0}{\Longrightarrow} s_1 \overset{d_1}{\Longrightarrow} \ldots$ we have $ExecTime(\pi) = \sum_{i \geqslant 0} d_i$

Semantics of TCTL

For time-divergent path $\pi \in s_0 \xRightarrow{d_0} s_1 \xRightarrow{d_1} \ldots$, we have:

$$\pi \models \Diamond^{J} \Psi \quad \text{iff} \quad \exists\, i \geqslant 0.\, s_i{+}d \models \Psi \text{ for some } d \in [0, d_i] \text{ with}$$

$$\sum_{k=0}^{i-1} d_k + d \in J \qquad \text{and}$$

where for $s_i = \langle \ell_i, \eta_i \rangle$ and $d \geqslant 0$ we have $s_i{+}d = \langle \ell_i, \eta_i{+}d \rangle$

TCTL-semantics for timed automata

- Let *TA* be a timed automaton with clocks C and locations *Loc*
- For TCTL-state-formula Φ, the *satisfaction set* *Sat*(Φ) is defined by:

$$\mathit{Sat}(\Phi) \;=\; \{\, s \in \mathit{Loc} \times \mathit{Eval}(C) \mid s \models \Phi \,\}$$

- *TA* satisfies TCTL-formula Φ iff Φ holds in all initial states of *TA*:

$$\mathit{TA} \models \Phi \quad \text{if and only if} \quad \forall \ell_0 \in \mathit{Loc}_0 .\, \langle \ell_0, \eta_0 \rangle \models \Phi$$

where $\eta_0(x) = 0$ for all $x \in C$

Characterizing timelock

- TCTL semantics is also well-defined for *TA* with timelock
- A state contains a timelock whenever no time-divergent paths emanate from it
- A state is *timelock-free* if and only if it satisfies $\exists\Box$true
  - some time-divergent path satisfies $\Box$true, i.e., there is $\geqslant 1$ time-divergent path
  - note: for fair CTL, the states in which a fair path starts also satisfy $\exists\Box$true
- *TA* is timelock-free iff $\forall s \in \mathit{Reach}(\mathit{TS}(\mathit{TA}))$: $s \models \exists\Box$true
- Timelocks can thus be checked by a timed CTL formula

TCTL model checking

- TCTL model-checking problem: $TA \models \Phi$ for non-Zeno TA

$$\underbrace{TA \models \Phi}_{\text{timed automaton}} \quad \text{iff} \quad \underbrace{TS(TA) \models \Phi}_{\text{infinite transition system}}$$

- Idea: consider a finite quotient of $TS(TA)$ wrt. a bisimulation
  - $TS(TA) / \cong$ is a *region* transition system and denoted $RTS(TA)$
  - dependence on Φ is ignored for the moment . . .
- Transform TCTL formula Φ into an "equivalent" CTL-formula $\widehat{\Phi}$
- Then: $TA \models_{\text{TCTL}} \Phi$ iff $\underbrace{RTS(TA)}_{\text{finite transition system}} \models_{\text{CTL}} \widehat{\Phi}$

Basic recipe of TCTL model checking

Input: timed automaton *TA* and TCTL formula Φ (both over *AP* and C)
Output: $TA \models \Phi$

$\widehat{\Phi} :=$ eliminate the timing parameters from Φ;

determine the equivalence classes under $\cong$;

construct the region transition system $TS = RTS(TA)$;

apply the CTL model-checking algorithm to check $TS \models \widehat{\Phi}$;

$TA \models \Phi$ if and only if $TS \models \widehat{\Phi}$

how does clock equivalence look like?

Eliminating timing parameters

- Eliminate all intervals $J \neq [0, \infty)$ from TCTL formulas
  - introduce a fresh clock, z say, that does not occur in *TA*
- Formally: for any state s of *TS*(*TA*) it holds:

$$s \models \exists\Diamond^{J}\Phi \quad \text{iff} \quad \underbrace{s\{z := 0\}}_{\text{state in } TS(TA \oplus z)} \models \exists\Diamond\big((z \in J) \wedge \Phi\big)$$

  - where $TA \oplus z$ is *TA* (over C) extended with $z \notin C$

atomic clock constraints are atomic propositions, i.e., a CTL formula results

Correctness

Let $\mathit{TA} = (\mathit{Loc}, \mathit{Act}, C, \hookrightarrow, \mathit{Loc}_0, \mathit{Inv}, \mathit{AP}, L)$. For clock $z \notin C$, let

$$\mathit{TA} \oplus z \;=\; (\mathit{Loc}, \mathit{Act}, C \cup \{ z \}, \hookrightarrow, \mathit{Loc}_0, \mathit{Inv}, \mathit{AP}, L).$$

For any state s of $\mathit{TS}(\mathit{TA})$ it holds that:

1. $s \models \exists \Diamond^{J} \Psi$ iff $\underbrace{s\{z := 0\}}_{\text{state in } \mathit{TS}(\mathit{TA} \oplus z)} \models \exists \Diamond \big((z \in J) \wedge \Psi \big)$

2. $s \models \forall \Diamond^{J} \Psi$ iff $\underbrace{s\{z := 0\}}_{\text{state in } \mathit{TS}(\mathit{TA} \oplus z)} \models \forall \Diamond \big((z \in J) \wedge \Psi \big)$

Clock equivalence $\cong$

(A) Equivalent clock valuations satisfy the same clock constraints g:

$$\eta \cong \eta' \;\Rightarrow\; (\eta \models g \quad \text{iff} \quad \eta' \models g)$$

(B) Time-divergent paths of equivalent states are “equivalent”

- this property guarantees that equivalent states satisfy the same path formulas

(C) The number of equivalence classes under $\cong$ is finite

Clock equivalence

- Correctness criteria (A) and (B) are ensured if equivalent states:
  - agree on the integer parts of all clock values, and
  - agree on the ordering of the fractional parts of all clocks

$\Rightarrow$ This yields a denumerable infinite set of equivalence classes

- Observe that:
  - if clocks exceed the maximal constant with which they are compared their precise value is not of interest

$\Rightarrow$ The number of equivalence classes is then finite (C)

Clock equivalence: definition

Clock valuations $\eta, \eta' \in \mathit{Eval}(C)$ are *equivalent*, denoted $\eta \cong \eta'$, if either:

- for all $x \in C$: $\eta(x) > c_x$ iff $\eta'(x) > c_x$, or
- for any $x, y \in C$ with $\eta(x) \leqslant c_x$ and $\eta(y) \leqslant c_y$ it holds:
  - $\lfloor \eta(x) \rfloor = \lfloor \eta'(x) \rfloor$ and $\mathit{frac}(\eta(x)) = 0$ iff $\mathit{frac}(\eta'(x)) = 0$, and
  - $\mathit{frac}(\eta(x)) \leqslant \mathit{frac}(\eta(y))$ iff $\mathit{frac}(\eta'(x)) \leqslant \mathit{frac}(\eta'(y))$.

$$s \cong s' \quad \text{iff} \quad \ell = \ell' \quad \text{and} \quad \eta \cong \eta'$$

Regions

- The *clock region* of $\eta \in \mathit{Eval}(C)$, denoted $[\eta]$, is defined by:

$$[\eta] \;=\; \{\, \eta' \in \mathit{Eval}(C) \mid \eta \cong \eta' \,\}$$

- The *state region* of $s = \langle \ell, \eta \rangle \in \mathit{TS}(\mathit{TA})$ is defined by:

$$[s] \;=\; \langle \ell, [\eta] \rangle \;=\; \{\, \langle \ell, \eta' \rangle \mid \eta' \in [\eta] \,\}$$

Bounds on the number of regions

The *number of clock regions* is bounded from below and above by:

$$|C|! * \prod_{x \in C} c_x \quad \leqslant \quad \underbrace{|\mathit{Eval}(C)/\cong|}_{\text{number of regions}} \quad \leqslant \quad |C|! * 2^{|C|-1} * \prod_{x \in C} (2c_x + 2)$$

where for the upper bound it is assumed that $c_x \geqslant 1$ for any $x \in C$

the number of state regions is $|\mathit{Loc}|$ times larger

Preservation of atomic properties

1. For $\eta, \eta' \in \mathit{Eval}(C)$ such that $\eta \cong \eta'$:

$$\eta \models g \quad \text{if and only if} \quad \eta' \models g \text{ for any } g \in \mathit{ACC}(\mathit{TA} \cup \Phi)$$

2. For $s, s' \in \mathit{TS}(\mathit{TA})$ such that $s \cong s'$:

$$s \models a \quad \text{if and only if} \quad s' \models a \text{ for any } a \in \mathit{AP}'$$

where AP' includes all propositions in *TA* and atomic clock constraints in *TA* and Φ

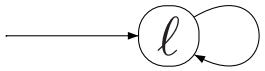
Clock equivalence is a bisimulation

Clock equivalence is a bisimulation equivalence over AP'

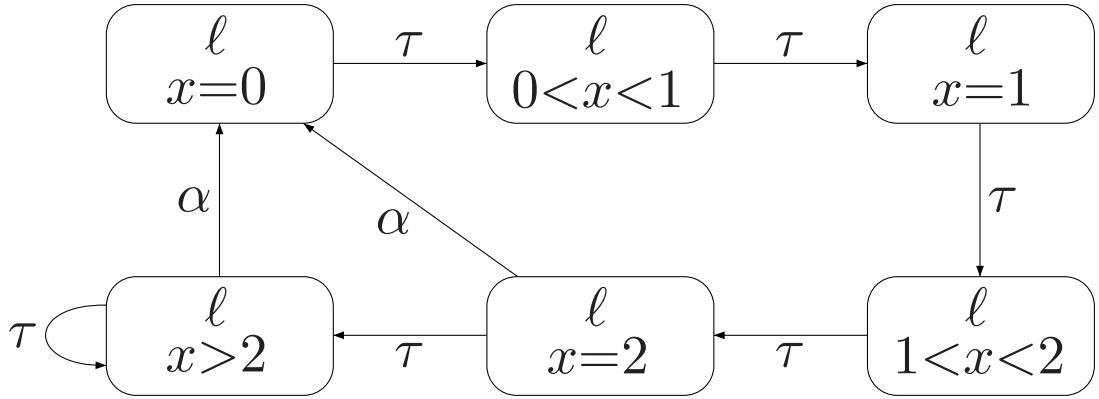
Region automaton: intuition

- Region automaton = quotient of $TS(TA)$ under $\cong$
- State regions are states in quotient transition system under $\cong$
- Transitions in region automaton "mimic" those in $TS(TA)$
- Delays are abstract
  - the exact delay is not recorded, only that some delay took place
  - if any clock x exceeds c_x, delays are self-loops
- Discrete transitions correspond to actions

A simple example



$x \geqslant 2 : \alpha$
$reset(x)$



Unbounded and successor regions

- Clock region $r_\infty = \{\, \eta \in \mathit{Eval}(C) \mid \forall x \in C.\, \eta(x) > c_x \,\}$ is *unbounded*
- r' is the *successor (clock) region* of r, denoted $r' = \mathit{succ}(r)$, if either:
  1. $r = r_\infty$ and $r = r'$, or
  2. $r \neq r_\infty$, $r \neq r'$ and $\forall \eta \in r$:

$$\exists d \in \mathbb{R}_{>0}.\ (\eta{+}d \in r' \quad \text{and} \quad \forall 0 \leqslant d' \leqslant d.\, \eta{+}d' \in r \cup r')$$

- The *successor region*: $\mathit{succ}(\langle \ell, r \rangle) = \langle \ell, \mathit{succ}(r) \rangle$
- Note: the location invariants are ignored so far!

Characterizing time convergence

For non-zeno *TA* and $\pi = s_0\, s_1\, s_2 \ldots$ a path in *TS*(*TA*):

(a) π is *time-convergent* $\Rightarrow$ $\exists$ state region $\langle \ell, r \rangle$ such that for some j:

$$s_i \in \langle \ell, r \rangle \text{ for all } i \geqslant j$$

(b) If $\exists$ state region $\langle \ell, r \rangle$ with $r \neq r_\infty$ and an index j such that:

$$s_i \in \langle \ell, r \rangle \text{ for all } i \geqslant j$$

then π is *time-convergent*

time-convergent paths are paths that only perform delays from some time instant on

Region automaton

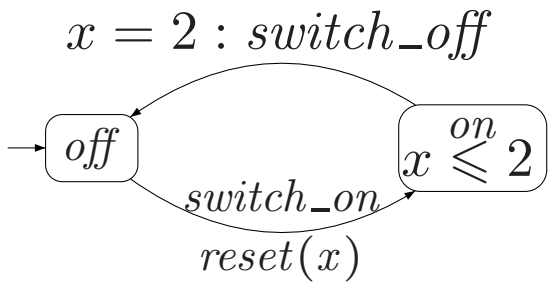
For non-zeno *TA* with $\mathit{TS}(\mathit{TA}) = (S, \mathit{Act}, \to, I, \mathit{AP}, L)$ let:

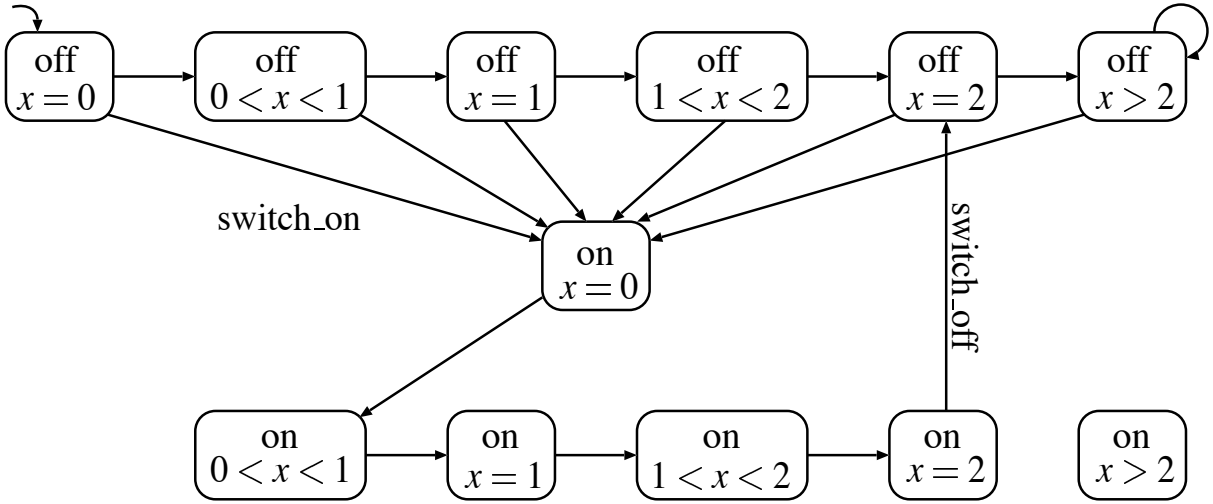
$$\mathit{RTS}(\mathit{TA}, \Phi) = (S', \mathit{Act} \cup \{\tau\}, \to', I, \mathit{AP}', L') \quad \text{with}$$

- $S' = S/\cong = \{[s] \mid s \in S\}$ and $I' = \{[s] \mid s \in I\}$, the state regions
- $L'(\langle \ell, r \rangle) = L(\ell) \cup \{ g \in \mathit{AP}' \setminus \mathit{AP} \mid r \models g \}$
- $\to'$ is defined by: $\dfrac{\ell \xrightarrow{g:\alpha,D} \ell' \quad r \models g \quad \text{reset } D \text{ in } r \models \mathit{Inv}(\ell')}{\langle \ell, r \rangle \xrightarrow{\alpha}' \langle \ell', \text{reset } D \text{ in } r \rangle}$

and $\dfrac{r \models \mathit{Inv}(\ell) \quad \mathit{succ}(r) \models \mathit{Inv}(\ell)}{\langle \ell, r \rangle \xrightarrow{\tau}' \langle \ell, \mathit{succ}(r) \rangle}$

Example: simple light switch





Correctness theorem [Alur and Dill, 1989]

For non-Zeno timed automaton TA and TCTL$_\Diamond$ formula Φ:

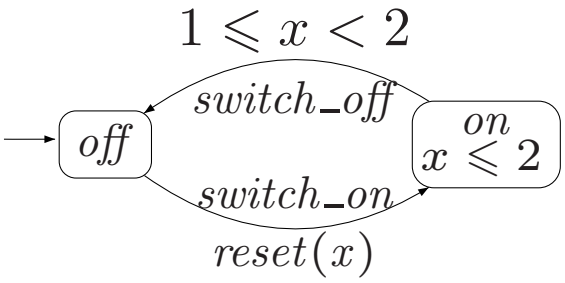
$$\underbrace{TA \models \Phi}_{\text{TCTL semantics}} \quad \text{iff} \quad \underbrace{RTS(TA, \Phi) \models \Phi}_{\text{CTL semantics}}$$

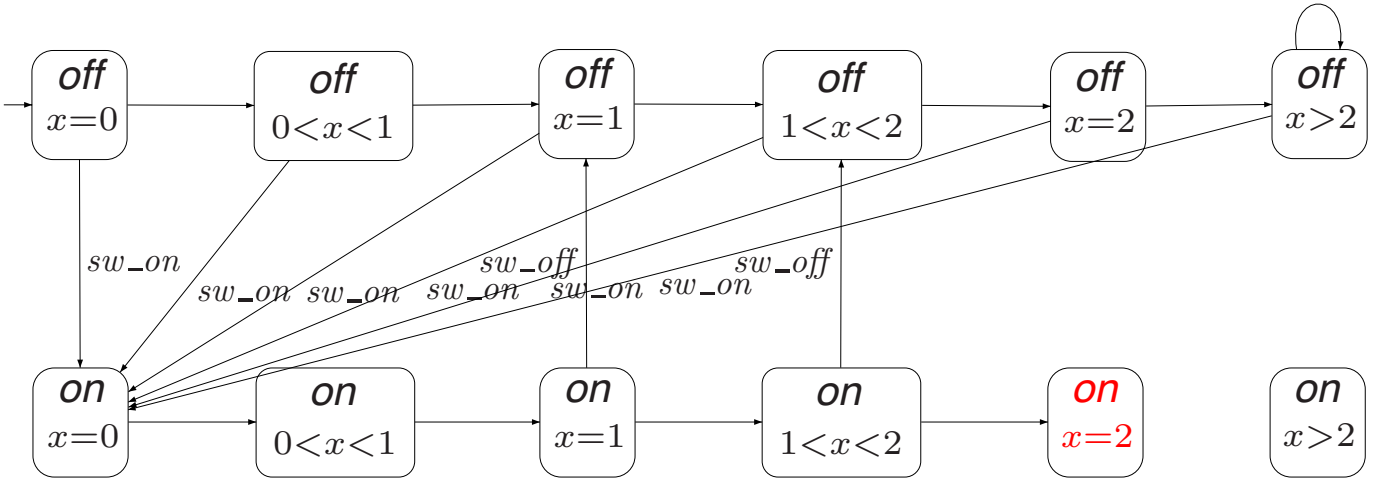
Characterizng timelock freedom

Non-zeno TA is timelock-free iff no reachable state in $RTS(TA)$ is terminal

timelocks can thus be checked by a reachability analysis of $RTS(TA)$

Example


$1 \leqslant x < 2$
switch_off
off
on
$x \leqslant 2$
switch_on
reset(x)


off
$x=0$
off
$0<x<1$
off
$x=1$
off
$1<x<2$
off
$x=2$
off
$x>2$
sw_on
sw_on
sw_on
sw_off
sw_on
sw_on
sw_off
sw_on
on
$x=0$
on
$0<x<1$
on
$x=1$
on
$1<x<2$
on
$x=2$
on
$x>2$

Time complexity

For timed automaton *TA* and TCTL$_\Diamond$ formula Φ, the model-checking problem

TA $\models \Phi$ can be determined in time $\mathcal{O}\left((N+K)\cdot \mid \Phi \mid\right)$,

where N and K are the number of states and transitions in *RTS*(*TA*, Φ)

Other verification problems

1. The TCTL model-checking problem is PSPACE-complete
2. Model checking safety, reachability, or ω-regular properties in TA is PSPACE-complete
3. Model checking LTL and CTL against TA is PSPACE-complete
4. The model-checking problem for timed LTL is undecidable
5. The satisfaction problem for TCTL is undecidable

all facts without proof