Model Checking I
alias
Reactive Systems Verification

Luca Tesei

MSc in Computer Science, University of Camerino

Topics

- Program Graphs
- Semantics of Program Graphs as Transition Systems

Material

Reading:

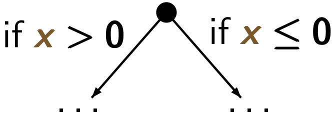
Chapter 2 of the book, pages 29–35.

More:

The slides in the following pages are taken from the material of the course “Introduction to Model Checking” held by Prof. Dr. Ir. Joost-Pieter Katoen at Aachen University.

Data-dependent systems

problem: TS-representation of conditional branchings **?**



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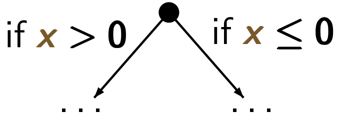

example: sequential program

```
WHILE $x > 0$ DO
    $x := x - 1$;
    $y := y + 1$
OD
...
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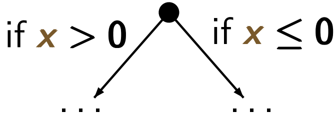

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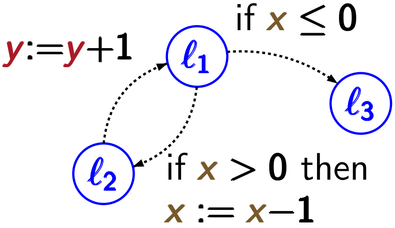

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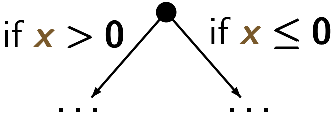

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if $x \leq 0$

$\dots$

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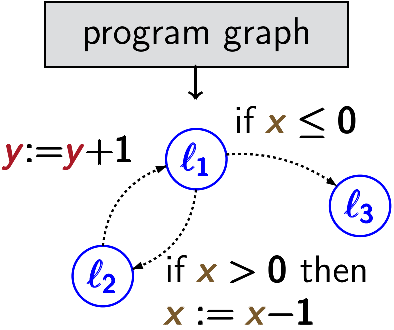

program graph

$y := y+1$

ℓ_1

if $x \leq 0$

ℓ_3

ℓ_2

if $x > 0$ then $x := x-1$

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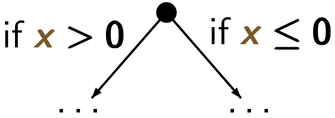

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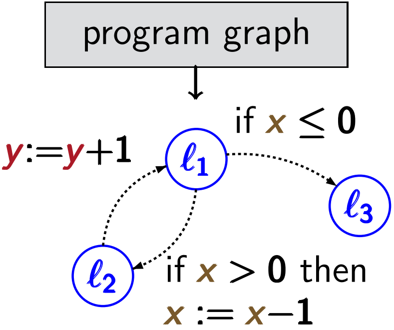
ℓ_1, ℓ_2, ℓ_3 are locations,
i.e., control states



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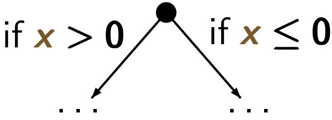

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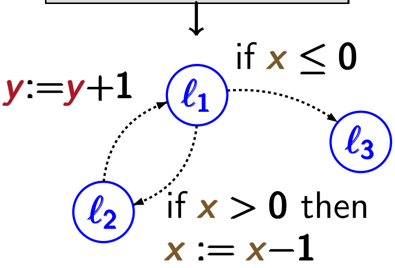

states of the transition system:

locations + relevant data (*here:* values for x and y)

Example: TS for sequential program

initially: $x = 2$, $y = 0$

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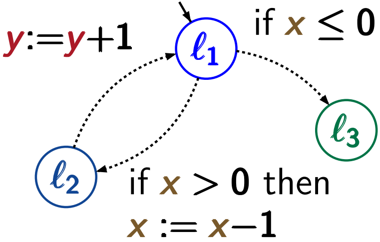

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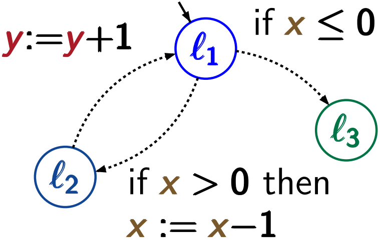
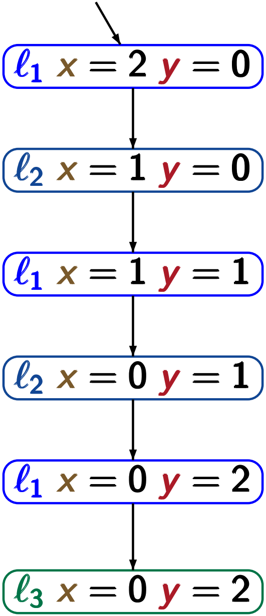




Example: TS for sequential program

initially: $x = 2$, $y = 0$

$\ell_1 \longrightarrow$ `WHILE` $x > 0$ `DO`

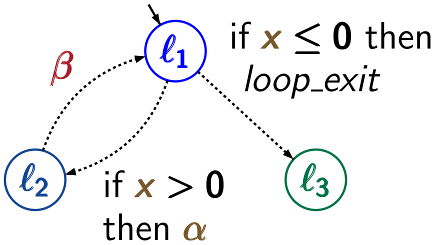
$\qquad x := x-1 \quad \longleftarrow$ $\boxed{\text{action } \alpha}$

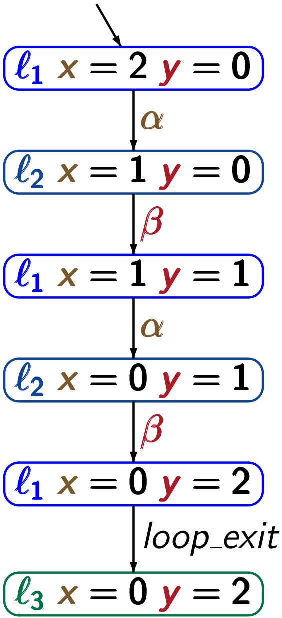
$\ell_2 \longrightarrow$ $\qquad y := y+1 \quad \longleftarrow$ $\boxed{\text{action } \beta}$

`OD`

$\ell_3 \longrightarrow$ $\ldots$

program graph





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Notation: $Eval(Var)$ = set of evaluations for Var

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$$\boxed{Cond(Var) = \text{ set of Boolean conditions on the variables in } Var}$$

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Example:

$$[x{=}0, y{=}3, z{=}6] \models \neg x \wedge y{<}z$$

$$[x{=}0, y{=}3, z{=}6] \not\models x \vee y{=}z$$

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if γ is "$(x, y) := (2x{+}y, 1{-}x)$" then:

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Program graph (PG)

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A *program graph* over Var is a tuple

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example: if α is the assignment $x:=x+y$ then

$$\mathit{Effect}(\alpha, [x=1, y=7]) = [x=8, y=7]$$

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  specifies conditional transitions of the form $\ell \overset{g:\alpha}{\hookrightarrow} \ell'$

ℓ, ℓ' are locations, $g \in \mathit{Cond}(\mathit{Var})$, $\alpha \in \mathit{Act}$

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TS-semantics of a program graph

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program graph $\mathcal{P}$ over Var

$$\Downarrow$$

transition system $\mathcal{T}_{\mathcal{P}}$

TS-semantics of a program graph

program graph $\mathcal{P}$ over ***Var***

$$\Downarrow$$

transition system $\mathcal{T}_{\mathcal{P}}$

states in $\mathcal{T}_{\mathcal{P}}$ have the form

$\langle \ell, \eta \rangle$

location

variable evaluation

TS-semantics of a program graph

Let $\mathcal{P} = (Loc, Act, Effect, \hookrightarrow, Loc_0, g_0)$ be a PG.
The transition system of $\mathcal{P}$ is:

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The transition relation $\longrightarrow$ is given by the following rule:

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Structured operational semantics (SOS)

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is a shortform notation in **SOS**-style.

$$\frac{premise}{conclusion}$$

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is a shortform notation in **SOS**-style.

It means that $\longrightarrow$ is the smallest relation such that:

$$\boxed{\text{if } \ell \xhookrightarrow{g:\alpha} \ell' \;\wedge\; \eta \models g \text{ then } \langle \ell, \eta \rangle \xrightarrow{\alpha} \langle \ell', Effect(\alpha, \eta) \rangle}$$

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- atomic propositions: $\mathit{AP} = \mathit{Loc} \cup \mathit{Cond}(\mathit{Var})$

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- labeling function:

$$L(\langle \ell, \eta \rangle) \ = \ \{\ell\} \cup \{g \in Cond(Var) : \eta \models g\}$$

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- initial states: $S_0 = \{\langle \ell, \eta \rangle : \ell \in Loc_0, \eta \models g_0\}$
- $\longrightarrow$ is given by the following rule:

$$\frac{\ell \overset{g:\alpha}{\hookrightarrow} \ell' \;\wedge\; \eta \models g}{\langle \ell, \eta \rangle \xrightarrow{\alpha} \langle \ell', Effect(\eta, \alpha) \rangle}$$

- atomic propositions: $AP = Loc \cup Cond(Var)$
- labeling function:

$$L(\langle \ell, \eta \rangle) \; = \; \{\ell\} \cup \{g \in Cond(Var) : \eta \models g\}$$